

BRAVE NEW MOTIVIC HOMOTOPY THEORY I

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ABSTRACT. This series of papers is dedicated to the study of motivic homotopy theory in the context of brave new or spectral algebraic geometry. In Part I we set up the theory and prove an analogue of the localization theorem of Morel–Voevodsky in classical motivic homotopy theory.

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1. INTRODUCTION

1.1. Brave new algebraic geometry.

1.1.1. *Brave new algebra* is a term coined by F. Waldhausen to describe the study of structured ring spectra and their invariants. In the brave new world, sets are replaced by spaces, abelian groups by spectra, and commutative rings by \mathcal{E}_{∞} -ring spectra; these are spectra equipped with a multiplicative structure that is associative and commutative only up to coherent homotopy.

For example, the Eilenberg–MacLane spectrum HR of any commutative ring R is an \mathcal{E}_{∞} -ring spectrum. The functor $R \mapsto HR$ is fully faithful, so we may view the category of commutative rings as a full subcategory of the category of \mathcal{E}_{∞} -ring spectra. However, general \mathcal{E}_{∞} -ring spectra are not $H\mathbf{Z}$ -algebras; the initial object is now the *sphere spectrum* \mathbf{S} . There are many interesting purely topological examples of algebras over \mathbf{S} , like the complex K-theory spectrum KU and the complex cobordism spectrum MU .

1.1.2. In algebraic geometry, commutative rings appear as rings of functions on schemes. Carrying further the point of view that \mathcal{E}_∞ -ring spectra are generalized commutative rings, it is natural to consider a brave new version of algebraic geometry, or “spectral algebraic geometry”¹, where \mathcal{E}_∞ -ring spectra are viewed as rings of functions on *spectral schemes*. Such a theory was developed by J. Lurie [Lur16b] and Toën–Vezzosi [TV08] in order to bring the power of the language of schemes and stacks to brave new algebra. For example, see [Lur09b] for a spectacular application to the theory of the ring spectrum of topological modular forms, and [CMNN16] for a beautiful application to the algebraic K-theory of \mathcal{E}_∞ -ring spectra.

1.1.3. A spectral scheme is, roughly speaking, a classical scheme S_{cl} together with a structure sheaf \mathcal{O}_S of connective \mathcal{E}_∞ -ring spectra, such that $\pi_0(\mathcal{O}_S) = \mathcal{O}_{S_{\text{cl}}}$. We view S as a sort of infinitesimal thickening of S_{cl} , in the same way that S_{cl} is a thickening of $(S_{\text{cl}})_{\text{red}}$.

Conversely, we may view any classical scheme as a spectral scheme with discrete structure sheaf, via the Eilenberg–MacLane functor $R \mapsto HR$.

We will review the theory of spectral schemes in some detail in Sect. 2, following the extensive work of Lurie and Toën–Vezzosi. We will see that it is possible to define well-behaved and useful notions of étale or smooth morphism of spectral schemes, but there some surprises. For instance, it turns out that the natural notion of smoothness in the brave new world does not agree with the classical notion. For example, a smooth homomorphism of commutative rings $A \rightarrow B$ may induce a morphism of \mathcal{E}_∞ -ring spectra $HA \rightarrow HB$ which is *not* smooth. Conversely, smooth morphisms of \mathcal{E}_∞ -ring spectra may not be flat.

Similarly, brave new vector bundles turn out to be different than classical vector bundles. For example, even the brave new affine line over $\text{Spec}(\mathbf{H}\mathbf{F}_p)$ is not the same as the affine line over $\text{Spec}(\mathbf{F}_p)$.

1.2. Motivic homotopy theory.

1.2.1. Motivic homotopy theory is an algebro-geometric analogue of the stable homotopy theory of topological spaces. It is the category of coefficients for generalized motivic cohomology theories of schemes, like (homotopy invariant) algebraic K-theory and algebraic cobordism, just as the stable homotopy category is the category of coefficients for generalized cohomology theories of spaces, like complex K-theory and complex cobordism. It was constructed by F. Morel and V. Voevodsky in [MV99], and its applications include the resolution of the Quillen–Lichtenbaum conjecture relating motivic cohomology and algebraic K-theory (see [Voe11], [SV00], [GL01]).

1.2.2. The subject of this paper is the construction of a brave new version of the motivic homotopy category. This provides a setting for the study of generalized motivic cohomology theories of connective \mathcal{E}_∞ -ring spectra.

For example, our construction gives rise to a notion of “brave new motivic homotopy groups” of connective \mathcal{E}_∞ -ring spectra. In a sequel we will consider a variant of algebraic K-theory of ring spectra which becomes representable in the brave new motivic homotopy category. An interesting question is whether it is possible to construct some version of motivic cohomology for connective \mathcal{E}_∞ -ring spectra that is representable as a brave new motivic spectrum.

¹The terms “spectral algebraic geometry”, “brave new algebraic geometry”, and “derived algebraic geometry” have all been used in the literature.

1.2.3. Recall that classically, motivic homotopy theory over a base scheme S is built from smooth S -schemes, imposing a Mayer–Vietoris excision property and homotopy invariance with respect to the affine line \mathbf{A}^1 .

In our setting, we can perform an analogous construction to obtain the brave new motivic homotopy category. Perhaps surprisingly, this appears to give something new even over classical base schemes.

1.2.4. Let S be a spectral scheme. We now give a more precise overview of the construction.

A *motivic space* over S will be a presheaf of spaces \mathcal{F} on the category of smooth spectral S -schemes, satisfying the properties of *Nisnevich excision* and *homotopy invariance* with respect to the brave new affine line.

1.2.5. Nisnevich excision can be formulated as follows.

Firstly, we impose the condition that the presheaf \mathcal{F} is *reduced*, i.e. it sends the empty scheme to a contractible space $\mathcal{F}(\emptyset)$ (see Paragraph C.2).

Then, we consider *Nisnevich squares*, i.e. homotopy cartesian squares of smooth spectral S -schemes

$$(1.1) \quad \begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

with j an open immersion and p étale, and the induced morphism $p^{-1}(X - U) \rightarrow X - U$ an isomorphism of underlying reduced classical schemes.

We require that the induced square of spaces

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{j^*} & \Gamma(U, \mathcal{F}) \\ \downarrow p^* & & \downarrow \\ \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(W, \mathcal{F}) \end{array}$$

is homotopy cartesian.

1.2.6. For the condition of homotopy invariance, we need to introduce the brave new affine line. This is the spectral scheme \mathbf{I} defined as $\mathrm{Spec}(\mathbf{S}\{t\})$, where $\mathbf{S}\{t\}$ is the free \mathcal{E}_∞ -algebra on one generator t over the sphere spectrum \mathbf{S} .

Then we say that a presheaf \mathcal{F} is *\mathbf{I} -homotopy invariant* if, for any smooth spectral S -scheme X , the canonical morphism of spaces

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(\mathbf{I} \times X, \mathcal{F})$$

is invertible.

1.2.7. Let $\mathrm{MotSpc}^{\mathcal{E}_\infty}(S)$ denote the $(\infty, 1)$ -category of motivic spaces over S . It is easy to see that the assignment $S \mapsto \mathrm{MotSpc}^{\mathcal{E}_\infty}(S)$ admits the following functorialities:

Given a morphism $f : T \rightarrow S$ of spectral schemes, there is an inverse image functor

$$f^* : \mathrm{MotSpc}^{\mathcal{E}_\infty}(S) \rightarrow \mathrm{MotSpc}^{\mathcal{E}_\infty}(T)$$

which is left adjoint to a direct image functor

$$f_* : \mathrm{MotSpc}^{\mathcal{E}_\infty}(T) \rightarrow \mathrm{MotSpc}^{\mathcal{E}_\infty}(S).$$

1.2.8. If f is smooth, there is an “extra” operation

$$f_{\#} : \text{MotSpc}^{\varepsilon_{\infty}}(\mathbf{T}) \rightarrow \text{MotSpc}^{\varepsilon_{\infty}}(\mathbf{S}),$$

left adjoint to f^* .

This operation is compatible with the operations $(f^*, f_*, \otimes, \underline{\text{Hom}})$ in the sense that it satisfies various base change and projection formulas; see Sect. 5.

1.3. Motivic spectra.

1.3.1. Let \mathbf{T}_S denote the Thom space of the vector bundle $S \times \mathbf{I}$ over S , i.e. the cofibre $(S \times \mathbf{I})/(S \times \mathbf{I}^{\times})$, where \mathbf{I}^{\times} denotes the complement of the zero section.

A *motivic spectrum* over S is the data of a sequence of pointed motivic spaces $(\mathcal{F}_n)_{n \geq 0}$ over S , and isomorphisms

$$\alpha_n : \mathcal{F}_n \xrightarrow{\sim} \Omega_{\mathbf{T}}(\mathcal{F}_{n+1})$$

for each integer $n \geq 0$. Here $\mathcal{F} \mapsto \Omega_{\mathbf{T}}(\mathcal{F})$ denotes the \mathbf{T}_S -loop space functor.

1.3.2. The category $\text{MotSpt}^{\varepsilon_{\infty}}(\mathbf{S})$ of motivic spectra can be described as the result of formally inverting the object \mathbf{T}_S with respect to the monoidal product. In particular, it admits a canonical symmetric monoidal structure, and there is a canonical symmetric monoidal functor

$$\Sigma_S^{\infty} : \text{MotSpc}^{\varepsilon_{\infty}}(\mathbf{S})_{\bullet} \rightarrow \text{MotSpt}^{\varepsilon_{\infty}}(\mathbf{S})$$

which has the universal property of being initial in the category of symmetric monoidal functors which send \mathbf{T}_S to an invertible object.

1.3.3. Our functorialities $(f^*, f_*, f_{\#})$ extend to the assignment $S \mapsto \text{MotSpt}^{\varepsilon_{\infty}}(\mathbf{S})$. Hence we have inverse image functors $f_{\text{MotSpt}^{\varepsilon_{\infty}}}^*$ for any morphism of schemes f , which admit right adjoints $f_{\text{MotSpt}^{\varepsilon_{\infty}}}^{\text{MotSpt}^{\varepsilon_{\infty}}}$ (resp. left adjoints $f_{\#}^{\text{MotSpt}^{\varepsilon_{\infty}}}$, when f is smooth).

1.3.4. Given a classical scheme S , we may view S as a spectral scheme and consider the category $\text{MotSpt}^{\varepsilon_{\infty}}(\mathbf{S})$. It is important to note that, in general, this does not coincide with the classical motivic homotopy category of Morel–Voevodsky. In characteristic zero they agree (see Theorem 3.7.2).

1.4. The localization theorem.

1.4.1. The main result of this paper can be stated as follows.

Theorem 1.4.2 (Localization). *Let $i : Z \hookrightarrow S$ be a closed immersion of spectral schemes with quasi-compact open complement $j : U \hookrightarrow S$. Then the following statements hold:*

(1) *For any motivic space \mathcal{F} over S , there is a canonical homotopy cocartesian square*

$$(1.2) \quad \begin{array}{ccc} j_{\#}j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_{\#}j^*(\text{pt}_S) & \longrightarrow & i_*i^*(\mathcal{F}), \end{array}$$

where pt_S denotes the terminal motivic space over S .

(2) *For any pointed motivic space (\mathcal{F}, x) over S , there is a canonical cofibre sequence*

$$(1.3) \quad j_{\#}j^*(\mathcal{F}, x) \rightarrow (\mathcal{F}, x) \rightarrow i_*i^*(\mathcal{F}, x).$$

(3) For any motivic spectrum \mathbb{E} over S , there is a canonical exact triangle

$$(1.4) \quad j_{\#}j^*(\mathbb{E}) \rightarrow \mathbb{E} \rightarrow i_*i^*(\mathbb{E}).$$

This is the combination of Theorem 6.2.6, Corollary 6.2.8, and Corollary 6.2.10.

1.4.3. In the setting of classical algebraic geometry, this theorem was demonstrated by Morel–Voevodsky (with some additional finiteness conditions on the schemes, see [MV99]). Our proof follows the same general strategy, but we make some parts of the argument more robust, so that it survives in the spectral setting.

If we take a closed immersion $i : Z \hookrightarrow S$ of classical schemes, our proof, when interpreted in the setting of classical algebraic geometry, gives a proof of Morel–Voevodsky’s theorem for classical schemes, without any finiteness assumptions.

One subtlety that appears when we remove finiteness hypotheses is that Čech descent (descent with respect to Čech covers) is no longer equivalent to hyperdescent (descent with respect to arbitrary hypercovers). We refer to [Lur09a, §6.5.4] and [Hoy15a, Appendix C] for an explanation of this distinction.

1.4.4. The localization theorem can be reformulated as follows:

Corollary 1.4.5. *Let $i : Z \hookrightarrow S$ be a closed immersion of spectral schemes with quasi-compact open complement $j : U \hookrightarrow S$. Then the direct image functor*

$$i_* : \mathrm{MotSpt}^{\mathcal{E}\infty}(Z) \rightarrow \mathrm{MotSpt}^{\mathcal{E}\infty}(S)$$

is fully faithful, and its essential image is spanned by motivic spectra supported on Z .

In the language of [BGT13], there is a short exact sequence of stable presentable $(\infty, 1)$ -categories

$$\mathrm{MotSpt}^{\mathcal{E}\infty}(Z) \xrightarrow{i_*} \mathrm{MotSpt}^{\mathcal{E}\infty}(S) \xrightarrow{j^*} \mathrm{MotSpt}^{\mathcal{E}\infty}(U).$$

1.4.6. Since the underlying classical scheme of a spectral scheme is a closed subscheme with empty complement, an immediate consequence of the localization theorem is that the motivic homotopy categories of a spectral scheme coincide with those of its underlying classical scheme.

Corollary 1.4.7 (Nilpotent invariance). *Let S be a spectral scheme, and write $i : S_{\mathrm{cl}} \hookrightarrow S$ for the inclusion of the underlying classical scheme. Then the adjunctions*

$$\begin{aligned} i^* : \mathrm{MotSpc}^{\mathcal{E}\infty}_{\bullet}(S) &\rightleftarrows \mathrm{MotSpc}^{\mathcal{E}\infty}_{\bullet}(S_{\mathrm{cl}}) : i_*, \\ i^* : \mathrm{MotSpt}^{\mathcal{E}\infty}(S) &\rightleftarrows \mathrm{MotSpt}^{\mathcal{E}\infty}(S_{\mathrm{cl}}) : i_* \end{aligned}$$

are equivalences of $(\infty, 1)$ -categories.

Here we can also take i to be the inclusion of the *reduced* closed subscheme of S_{cl} .

1.4.8. We emphasize that, despite nilpotent invariance, the category $\mathrm{MotSpt}^{\mathcal{E}\infty}(S_{\mathrm{cl}})$ does *not* appear to coincide with the classical motivic homotopy category $\mathrm{MotSpt}^{\mathrm{cl}}(S_{\mathrm{cl}})$ in general.

In characteristic zero, this does hold:

Theorem 1.4.9. *Let S be a spectral scheme of characteristic zero. Then there are canonical equivalences of $(\infty, 1)$ -categories*

$$\begin{aligned} \mathrm{MotSpc}^{\mathcal{E}\infty}(S) &= \mathrm{MotSpc}^{\mathrm{cl}}(S_{\mathrm{cl}}), \\ \mathrm{MotSpt}^{\mathcal{E}\infty}(S) &= \mathrm{MotSpt}^{\mathrm{cl}}(S_{\mathrm{cl}}). \end{aligned}$$

This theorem is the combination of nilpotent invariance and Theorem 3.7.2.

1.5. Other homotopical algebraic geometry contexts. By considering various versions of “homotopical commutative rings”, we obtain many variants of homotopical algebraic geometry which each have a unique flavour.

1.5.1. In [Kha16] we developed motivic homotopy theory in the setting of *derived* algebraic geometry, where schemes are locally modelled on *simplicial commutative rings*. This is essentially the minimal extension of classical algebraic geometry where derived fibred products exist. Every simplicial commutative ring gives rise to a connective \mathcal{E}_∞ -ring spectrum, but it will still be \mathbf{HZ} -linear, so we do not see any of the purely topological phenomena in brave new algebra. Furthermore, the notions of smoothness and of vector bundles in this setting agree with the usual notions from classical algebraic geometry. In particular, the motivic homotopy theory of a classical scheme, when viewed as a derived scheme, coincides with the classical Morel–Voevodsky construction. Hence this is a rather conservative version of homotopical algebraic geometry, which is very close to classical algebraic geometry. On the other hand, it should be the correct setting to study classical algebro-geometric phenomena like algebraic cycles.

1.5.2. It is also possible to work in the setting of \mathcal{E}_n -algebraic geometry, where $n \geq 2$. Here our “homotopical commutative rings” are connective \mathcal{E}_n -ring spectra. Recall that \mathcal{E}_n -ring spectra are “less” commutative than \mathcal{E}_∞ -ring spectra; the \mathcal{E}_∞ -operad is the homotopy colimit of the \mathcal{E}_n -operads as n increases. The localization theorem continues to hold in this setting, with the same proof as in this paper. We find this interesting, because for any field k , we obtain infinitely many versions of the motivic homotopy category,

$$\mathrm{MotSpt}^{\mathcal{E}_2}(k), \mathrm{MotSpt}^{\mathcal{E}_3}(k), \dots, \mathrm{MotSpt}^{\mathcal{E}_\infty}(k),$$

which seem to be different unless k is of characteristic zero.

The main reason we have decided to consider only the case $n = \infty$ in this paper is that we could not find any references for many basic facts in \mathcal{E}_n -algebraic geometry that we need. We may return to this topic in the future.

1.6. Contents. In Sect. 2 we review the basics of spectral algebraic geometry. We follow the “functor of points” approach as in [TV08] or [AG14]; an alternative approach using locally ringed ∞ -toposes can be found in [Lur16b]. There is nothing new in this section, with the exception of Paragraph 2.13, which we could not find a reference for.

In Sect. 3 we give the construction of the brave new motivic homotopy category over a spectral scheme. In Sects. 4–6 we discuss the functorialities that the assignment $S \mapsto \mathrm{MotSpt}^{\mathcal{E}_\infty}(S)$ is equipped with, and verify some useful base change and projection formulas.

In Sect. 7 we prove the main result of this paper, the localization theorem.

Appendix A is dedicated to some generalities about Morel–Voevodsky homotopy theory in abstract settings. There is no original mathematics in this section.

Appendix B introduces an axiomatic framework which we find convenient for discussing base change and projection formulas in abstract categories of coefficients.

Appendix C contains some technical topos-theoretic lemmas that we use in Sect. 6 to prove that the functor i_* of direct image along a closed immersion commutes with contractible colimits.

1.7. Conventions and notation. We will use the language of $(\infty, 1)$ -categories freely throughout the text. Though we will use the language in a model-independent way, we fix for concreteness the model of quasi-categories as developed by A. Joyal and J. Lurie. Our main references are [Lur09a] and [Lur16a].

All constructions will be of homotopy invariant nature by default. For example, the term “category” will mean “ $(\infty, 1)$ -category” (= quasi-category); when we want to refer to an ordinary category, we will use the term “ordinary category”. Similarly, (co)limits will be homotopy (co)limits, (co)cartesian squares will be homotopy (co)cartesian, etc.

\mathbf{Spc} will denote the $(\infty, 1)$ -category of spaces (= ∞ -groupoids). In any $(\infty, 1)$ -category \mathbf{C} , we will write $\mathrm{Maps}_{\mathbf{C}}(x, y) \in \mathbf{Spc}$ for the space of morphisms from an object x to an object y .

We will say that a morphism in an $(\infty, 1)$ -category is *invertible* or an *isomorphism* where [Lur09a] uses the term *equivalence*. We will use the symbol “=” for isomorphic objects in an $(\infty, 1)$ -category.

A *left localization* of an $(\infty, 1)$ -category \mathbf{C} is a functor $\gamma : \mathbf{C} \rightarrow \mathbf{D}$ admitting a fully faithful right adjoint.

We write $\mathrm{PSh}(\mathbf{C})$ for the $(\infty, 1)$ -category of presheaves (of spaces) on an $(\infty, 1)$ -category \mathbf{C} , i.e. functors $(\mathbf{C})^{\mathrm{op}} \rightarrow \mathbf{Spc}$. Following Joyal [Joy04], we use the term ∞ -arena for accessible left localizations of $(\infty, 1)$ -categories of the form $\mathrm{PSh}(\mathbf{C})$ for \mathbf{C} essentially small; the term *presentable ∞ -category* is used in [Lur09a]. A morphism of ∞ -arenas is a colimit-preserving functor between ∞ -arenas.

In a pointed $(\infty, 1)$ -category \mathbf{C} , we write $\Sigma_{\mathbf{S}^1}$ and $\Omega_{\mathbf{S}^1}$ for the \mathbf{S}^1 -suspension and \mathbf{S}^1 -loop space functors, respectively.

We will write \mathbf{Spt} for the stable $(\infty, 1)$ -category of spectra.

We will say that a small $(\infty, 1)$ -category is *contractible* if its underlying space is (weakly) contractible.

Indexing categories for limits and colimits will always be assumed to be small.

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2. SPECTRAL ALGEBRAIC GEOMETRY

In this section we review the theory of spectral algebraic geometry. All the material covered is standard and can be found in [Lur16b] and [TV08], with the exception of Proposition 2.13.2.

2.1. Spectral prestacks.

2.1.1. Let $\mathcal{E}_{\infty}\text{-Ring}_{\mathrm{cn}}$ denote the category of connective \mathcal{E}_{∞} -ring spectra.

The Eilenberg–MacLane functor $R \mapsto HR$ defines a fully faithful functor $\mathrm{CRing} \rightarrow \mathcal{E}_{\infty}\text{-Ring}_{\mathrm{cn}}$, whose essential image is spanned by the discrete (= 0-truncated) connective \mathcal{E}_{∞} -ring spectra.

2.1.2. A *spectral prestack* is a presheaf of spaces on the category $(\mathcal{E}_{\infty}\text{-Ring}_{\mathrm{cn}})^{\mathrm{op}}$, i.e. a functor $\mathcal{E}_{\infty}\text{-Ring}_{\mathrm{cn}} \rightarrow \mathbf{Spc}$.

We write Prestk for the category of spectral prestacks.

2.1.3. Given a connective \mathcal{E}_{∞} -ring spectrum A , we will write $\mathrm{Spec}(A)$ for the spectral prestack represented by A . We say that a spectral prestack is an *affine spectral scheme* if it is represented by a connective \mathcal{E}_{∞} -ring spectrum, and write $\mathrm{Sch}_{\mathrm{aff}}$ for the full subcategory of Prestk spanned by affine spectral schemes.

Let S be a spectral prestack. For a connective \mathcal{E}_{∞} -ring spectrum A , we say that an *A-point of S* is a morphism $s : \mathrm{Spec}(A) \rightarrow S$, or equivalently a point of the space $S(A)$.

2.1.4. A *classical² prestack* is a presheaf on the opposite of the ordinary category of commutative rings.

Given a spectral prestack S , let S_{cl} denote its *underlying classical prestack*, defined as the restriction to ordinary commutative rings. The functor $S \mapsto S_{\text{cl}}$ admits a fully faithful left adjoint, embedding the category of classical prestacks as a full subcategory of spectral prestacks.

We refer to spectral prestacks of the form $\text{Spec}(A)$, with A an ordinary commutative ring, as *classical affine schemes*. The functor $S \mapsto S_{\text{cl}}$ sends spectral affine schemes to classical affine spectral schemes: we have $\text{Spec}(A)_{\text{cl}} = \text{Spec}(\pi_0(A))$ for any connective \mathcal{E}_∞ -ring spectrum A .

2.2. Quasi-coherent sheaves.

2.2.1. Let $S = \text{Spec}(A)$ be an affine spectral scheme. A *quasi-coherent module* on S is the datum of an A -module. We write $\mathcal{O}_{\text{Spec}(A)}$ for the quasi-coherent module given by A , viewed as a module over itself.

2.2.2. Let S be a spectral prestack. A *quasi-coherent \mathcal{O}_S -module* consists of the following data:

- (1) For every affine spectral scheme $\text{Spec}(A)$ and every morphism $s : \text{Spec}(A) \rightarrow S$, a quasi-coherent module \mathcal{F}_s on $\mathcal{O}_{\text{Spec}(A)}$.
- (2) For every pair of morphisms $s : \text{Spec}(A) \rightarrow S$, $s' : \text{Spec}(B) \rightarrow S$ fitting into a commutative triangle

$$\begin{array}{ccc} \text{Spec}(A) & \xrightarrow{s} & S, \\ \downarrow f & \nearrow s' & \\ \text{Spec}(B) & & \end{array}$$

an isomorphism $f^*(\mathcal{F}_{s'}) \xrightarrow{\sim} \mathcal{F}_s$.

- (3) A homotopy coherent system of compatibilities between all such isomorphisms.

2.2.3. More precisely, we define the category $\text{Qcoh}(S)$ as the limit

$$\text{Qcoh}(S) := \varprojlim_{\text{Spec}(A) \rightarrow S} \text{Qcoh}(\text{Spec}(A))$$

in the category of arenas.

This category is *stable*, as the property of stability is stable under limits of $(\infty, 1)$ -categories.

Better yet, we can define a presheaf of symmetric monoidal arenas $S \mapsto \text{Qcoh}(S)$ as the right Kan extension of the presheaf $A \mapsto A\text{-mod}$ along the Yoneda embedding $(\mathcal{E}_\infty\text{-Ring}_{\text{cn}})^{\text{op}} \rightarrow \text{Prestk}$.

In particular, for each morphism of spectral prestacks f , we have a symmetric monoidal colimit-preserving functor f^* , the inverse image functor, and its right adjoint f_* , the direct image functor.

²The adjective *classical* refers to the fact that they are defined on non-derived objects (ordinary commutative rings, not \mathcal{E}_∞ -ring spectra). In the literature they have been studied by C. Simpson and others under the name *higher prestacks*, since they may take values in arbitrary spaces, not just groupoids. In our terminology, prestacks are “higher” by default, and “non-higher” prestacks are *1-truncated* prestacks.

2.2.4. Let S be a spectral prestack. We write \mathcal{O}_S for the quasi-coherent module defined by $\mathcal{O}_{S,s} = \mathcal{O}_{\mathrm{Spec}(A)}$ for each affine spectral scheme $\mathrm{Spec}(A)$ and each A -point $s : \mathrm{Spec}(A) \rightarrow S$. This is the unit of the symmetric monoidal structure.

Given a quasi-coherent module \mathcal{F} on S , we write $\Gamma(X, \mathcal{F})$ for the space of sections over a spectral S -scheme X . This is by definition the mapping space $\mathrm{Maps}(\mathcal{O}_X, p^*(\mathcal{F}))$, where $p : X \rightarrow S$ is the structural morphism.

2.3. Spectral stacks.

2.3.1. Let $f : T \rightarrow S$ be a morphism of affine spectral schemes, $S = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(B)$.

Definition 2.3.2. (i) *The morphism f is locally of finite presentation if B is compact in the category of A -algebras.*

(ii) *The morphism f is flat if the functor $f^* : \mathrm{Qcoh}(S) \rightarrow \mathrm{Qcoh}(T)$ is exact. Equivalently, the morphism $f_{\mathrm{cl}} : \mathrm{Spec}(\pi_0(B)) \rightarrow \mathrm{Spec}(\pi_0(A))$ of underlying classical schemes is flat (i.e. $\pi_0(B)$ is flat, in the usual sense, as an $\pi_0(A)$ -module), and the canonical morphism $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_i(B)$ is invertible for each i .*

(iii) *The morphism f is an open immersion if it is locally of finite presentation, flat, and a monomorphism³. Equivalently, it is flat and the morphism $f_{\mathrm{cl}} : \mathrm{Spec}(\pi_0(B)) \rightarrow \mathrm{Spec}(\pi_0(A))$ of underlying classical schemes is an open immersion (in the classical sense).*

Remark 2.3.3. We will use the following observation repeatedly. Let $f : T \rightarrow S$ be a morphism of affine spectral schemes. If f is flat and S is classical, then T is also classical.

Similarly, if $f : T \rightarrow S$ is a flat morphism of affine spectral schemes, then the commutative square

$$\begin{array}{ccc} T_{\mathrm{cl}} & \hookrightarrow & T \\ \downarrow f_{\mathrm{cl}} & & \downarrow f \\ S_{\mathrm{cl}} & \hookrightarrow & S \end{array}$$

is cartesian.

2.3.4. The Zariski topology on $\mathrm{Sch}_{\mathrm{aff}}$ is the Grothendieck topology associated to the following pretopology. A family of morphisms of affine spectral schemes $(j_\alpha : U_\alpha \rightarrow X)_{\alpha \in \Lambda}$ is *Zariski covering* if and only if each j_α is an open immersion, and the family of functors $(j_\alpha)^* : \mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}(U_\alpha)$ is conservative.

Definition 2.3.5. A spectral stack is a spectral prestack satisfying descent with respect to the Zariski topology.

We write Stk for the full subcategory of Prestk spanned by spectral stacks.

2.4. Spectral schemes.

2.4.1. Let $j : U \rightarrow S$ be a morphism of spectral stacks.

Definition 2.4.2. (i) *If S is affine, then j is an open immersion if it is a monomorphism, and there exists a family $(j_\alpha : U_\alpha \rightarrow S)_\alpha$, with each j_α an open immersion of affine spectral schemes Definition 2.3.2 that factors through U and induces an effective epimorphism $\sqcup_\alpha U_\alpha \rightarrow U$.*

(ii) *For general S , the morphism j is an open immersion if for each connective \mathcal{E}_∞ -ring spectrum A and each A -point $s : \mathrm{Spec}(A) \rightarrow S$, the base change $U \times_S \mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A)$ is an open immersion in the sense of (i).*

³I.e. the canonical morphism $T \rightarrow T \times_S T$ is invertible.

2.4.3. Let S be a spectral stack. A *Zariski cover* of S is a small family of open immersions of spectral stacks $(j_\alpha : U_\alpha \rightarrow S)_\alpha$ such that the canonical morphism $\sqcup_\alpha U_\alpha \rightarrow S$ is surjective (i.e. an effective epimorphism in the topos of spectral stacks). If each U_α is an affine spectral scheme, we call this an *affine Zariski cover*.

We define:

Definition 2.4.4. A spectral scheme is a spectral stack S which admits an affine Zariski cover.

We write Sch for the full subcategory of Stk spanned by spectral schemes. It is closed under coproducts and fibred products.

Definition 2.4.5. (i) A spectral scheme S is quasi-compact if for any Zariski cover $(j_\alpha : U_\alpha \rightarrow S)_{\alpha \in \Lambda}$, there exists a finite subset $\Lambda_0 \subset \Lambda$ such that the family $(j_\alpha)_{\alpha \in \Lambda_0}$ is still a Zariski cover.

(ii) A morphism of spectral schemes $f : T \rightarrow S$ is quasi-compact if for any connective \mathcal{E}_∞ -ring spectrum A and any A -point $s : \text{Spec}(A) \rightarrow S$, the spectral scheme $T \times_S \text{Spec}(A)$ is quasi-compact.

(iii) A spectral scheme S is quasi-separated if for any open immersions $U \hookrightarrow S$ and $V \hookrightarrow S$, with U and V affine, the intersection $U \times_S V$ is quasi-compact.

2.4.6. We define a *classical scheme* to be a Zariski sheaf of *sets* on the category $(\text{CRing})^{\text{op}}$, admitting a Zariski affine cover. This is equivalent to the definition of scheme given in [GD71].

Given a spectral scheme S , the underlying classical prestack S_{cl} takes values in sets, and is a classical scheme. We therefore refer to S_{cl} as the *underlying classical scheme* of S .

2.5. Closed immersions.

2.5.1. Let $f : Y \rightarrow X$ be a morphism of spectral schemes. We say that the morphism f is *affine* if, for any connective \mathcal{E}_∞ -ring spectrum A and A -point $x : \text{Spec}(A) \rightarrow X$, the base change $Y \times_X \text{Spec}(A)$ is an affine spectral scheme.

2.5.2. If $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine, a morphism $i : Y \rightarrow X$ is a *closed immersion* if the homomorphism $A \rightarrow B$ induces a surjection $\pi_0(A) \rightarrow \pi_0(B)$.

In general a morphism $i : Y \rightarrow X$ is a *closed immersion* if it is affine, and for any connective \mathcal{E}_∞ -ring spectrum A and A -point $x : \text{Spec}(A) \rightarrow X$, the base change $Y \times_X \text{Spec}(A) \rightarrow \text{Spec}(A)$ is a closed immersion of affine spectral schemes.

Equivalently, i is a closed immersion if and only if it induces a closed immersion on underlying classical schemes.

2.5.3. A *nil-immersion* is a closed immersion $i : Y \hookrightarrow X$ which induces an isomorphism $Y_{\text{cl}} \rightarrow X_{\text{cl}}$ on underlying classical schemes.

2.5.4. Let $i : Z \hookrightarrow S$ be a closed immersion of spectral schemes. Let U be the spectral prestack defined as follows: for a connective \mathcal{E}_∞ -ring spectrum A , its A -points are A -points $s : \text{Spec}(A) \rightarrow S$ such that the base change $\text{Spec}(A) \times_S Z$ is the empty spectral scheme. One can show that U is a spectral scheme, and that the canonical morphism $U \rightarrow S$ is an open immersion.

We call $j : U \hookrightarrow S$ the *complementary open immersion* to i .

2.6. Vector bundles.

2.6.1. Just as in Paragraph 2.2, we can define a notion of *quasi-coherent algebra* on a spectral prestack S , such that the category of quasi-coherent algebras on $\mathrm{Spec}(A)$ coincides with the category of A -algebras.

2.6.2. Let S be a spectral scheme and \mathcal{A} a quasi-coherent algebra on S . Consider the presheaf $\mathrm{Spec}_S(\mathcal{A})$ on the category of spectral schemes over S , which sends a spectral S -scheme X with structural morphism f to the space of quasi-coherent algebra homomorphisms $\mathrm{Maps}(f^*(\mathcal{A}), \mathcal{O}_X)$.

The presheaf $\mathrm{Spec}_S(\mathcal{A})$ clearly satisfies Zariski descent. Hence it defines a spectral stack over S (there is a canonical equivalence $\mathrm{Sh}(\mathrm{Sch}/_S) = \mathrm{Sh}(\mathrm{Sch})/_S = \mathrm{Sh}(\mathrm{Sch}_{\mathrm{aff}})/_S$), which we call the *relative spectrum* of the quasi-coherent algebra \mathcal{A} .

Further, we have:

Lemma 2.6.3. *Let \mathcal{A} be a connective quasi-coherent algebra over a spectral scheme S . Then the spectral stack $\mathrm{Spec}_S(\mathcal{A})$ is a spectral scheme.*

This follows from functoriality in S , and the fact that for $S = \mathrm{Spec}(A)$ affine, we have $\mathrm{Spec}_S(\mathcal{A}) = \mathrm{Spec}(\Gamma(S, \mathcal{A}))$.

2.6.4. For a connective \mathcal{E}_∞ -ring spectrum A and an A -module M , we write $\mathrm{Free}_A(M)$ for the free A -algebra generated by M . The assignment $M \mapsto \mathrm{Free}_A(M)$ defines a functor, left adjoint to the forgetful functor from A -algebras to A -modules, so that there are canonical isomorphisms

$$\mathrm{Maps}_{A\text{-alg}}(\mathrm{Free}_A(M), B) \xrightarrow{\sim} \mathrm{Maps}_{A\text{-mod}}(M, B)$$

bifunctorial in M and B .

2.6.5. Let \mathcal{F} be a quasi-coherent module on S . The quasi-coherent algebra $\mathrm{Free}_{\mathcal{O}_S}(\mathcal{F})$ is defined by $\mathrm{Free}_{\mathcal{O}_S}(\mathcal{F})_s := \mathrm{Free}_{\mathcal{O}_{S,s}}(\mathcal{F}_s)$ for each connective \mathcal{E}_∞ -ring spectrum A and A -point $s : \mathrm{Spec}(A) \rightarrow S$.

2.6.6. A connective quasi-coherent module \mathcal{F} on S is *locally free* of rank n if there exists a Zariski cover $(j_\alpha : U_\alpha \rightarrow S)_\alpha$ such that each inverse image $j_\alpha^*(\mathcal{F})$ is a free quasi-coherent \mathcal{O}_{S_α} -module of rank n , i.e. $j_\alpha^*(\mathcal{F}) = \mathcal{O}_{S_\alpha}^{\oplus n}$.

Given a locally free module \mathcal{F} of finite rank, we define:

Definition 2.6.7. *The vector bundle associated to \mathcal{F} is the spectral S -scheme $S\{\mathcal{F}\} := \mathrm{Spec}_S(\mathrm{Free}_{\mathcal{O}_S}(\mathcal{F}^\vee))$.*

Note that any global section $s \in \Gamma(S, \mathcal{F})$ defines a section $s : S \hookrightarrow S\{\mathcal{F}\}$ of the structural morphism, which is a closed immersion. In particular, every vector bundle admits a zero section.

2.6.8. For an integer $n \geq 0$, we define the *affine space* of dimension n over a spectral scheme S , to be the total space of the free \mathcal{O}_S -module $\mathcal{O}_S^{\oplus n}$:

$$S\{t_1, \dots, t_n\} := S\{\mathcal{O}_S^{\oplus n}\} = \mathrm{Spec}_S(\mathrm{Free}_{\mathcal{O}_S}(\mathcal{O}_S^{\oplus n})).$$

Note that, for any morphism of spectral schemes $f : T \rightarrow S$, we have $S\{t_1, \dots, t_n\} \times_S T = T\{t_1, \dots, t_n\}$.

The underlying classical scheme of $S\{t_1, \dots, t_n\}$ is the classical affine space $S_{\mathrm{cl}}[t_1, \dots, t_n] = S_{\mathrm{cl}} \times \mathrm{Spec}(\mathbf{Z}[t_1, \dots, t_n])$. We note however that, even when S is classical, the spectral scheme $S\{t_1, \dots, t_n\}$ will not be classical, except in characteristic zero. We will look at another version of affine space in Paragraph 2.7 that is flat, and does agree with the classical affine space.

2.6.9. The *affine line* $S\{t\}$ over S is the affine space of dimension 1. Since the quasi-coherent module \mathcal{O}_S has a unit section (being a quasi-coherent *algebra*), the affine line admits both a zero and a unit section.

2.7. Flat affine spaces.

2.7.1. Given a connective \mathcal{E}_∞ -ring spectrum R , let $R[t_1, \dots, t_n]$ denote the polynomial R -algebra in n variables ($n \geq 1$).

This is by definition the monoid algebra $R[\mathbf{N}^n] = R \otimes \Sigma^\infty(\mathbf{N}^n)_+$, where \mathbf{N} is the set of natural numbers, viewed as a discrete \mathcal{E}_∞ -monoid. The underlying spectrum of $R[t_1, \dots, t_n]$ is the direct sum $\bigoplus_{(k_1, \dots, k_n) \in \mathbf{N}^n} R$.

We have $\pi_0(R[t_1, \dots, t_n]) = \pi_0(R)[t_1, \dots, t_n]$ for each n .

For an ordinary commutative ring R , we have $(HR)[t_1, \dots, t_n] = H(R[t_1, \dots, t_n])$, where H denotes the Eilenberg–MacLane functor.

2.7.2. For a spectral scheme S , let $S[t_1, \dots, t_n]$ denote the spectral scheme $S \times \text{Spec}(\mathbf{S}[t_1, \dots, t_n])$.

The underlying classical scheme $(S[t_1, \dots, t_n])_{\text{cl}}$ coincides with the classical affine space $S_{\text{cl}}[t_1, \dots, t_n]$ over S_{cl} .

For a classical scheme S , the spectral scheme $S[t_1, \dots, t_n]$ coincides with the classical affine space over S .

2.7.3. There is a canonical morphism of \mathcal{E}_∞ -ring spectra

$$(2.1) \quad R\{t_1, \dots, t_n\} \rightarrow R[t_1, \dots, t_n]$$

which corresponds by adjunction to the inclusions $\varphi_i : R \rightarrow R[t_1, \dots, t_n]$ ($1 \leq i \leq n$) into the component $(k_1, \dots, k_n) = (0, \dots, 1, \dots, 0)$ (only $k_i = 1$) of the direct sum $\bigoplus_{(k_1, \dots, k_n) \in \mathbf{N}^n} R$.

For a spectral scheme S , this gives rise to canonical morphisms

$$(2.2) \quad S[t_1, \dots, t_n] \rightarrow S\{t_1, \dots, t_n\}$$

for each $n \geq 0$.

We have:

Proposition 2.7.4. *Let S be a spectral scheme of characteristic zero⁴. Then the canonical morphism (2.2) is invertible for each $n \geq 0$.*

Proof. This follows for example from [Lur16a, Prop. 7.1.4.20]. □

2.8. Infinitesimal extensions.

2.8.1. Let $p : Y \rightarrow X$ be a morphism of affine spectral schemes, $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. Given a connective quasi-coherent module \mathcal{F} on Y , we let

$$Y \hookrightarrow Y_{\mathcal{F}} := \text{Spec}(B \oplus M)$$

denote the *trivial infinitesimal extension of Y along \mathcal{F}* , where $M = \Gamma(Y, \mathcal{F})$.

The morphism $Y \hookrightarrow Y_{\mathcal{F}}$ is the closed immersion induced by the homomorphism $B \oplus M \rightarrow B$, $(b, m) \mapsto b$.

2.8.2. A *derivation* of Y over X with values in \mathcal{F} , is a retraction of the morphism $Y \hookrightarrow Y_{\mathcal{F}}$ (in the category of affine spectral schemes over X). There is a canonical retraction, the *trivial derivation* d_{triv} , defined by the morphism $B \rightarrow B \oplus M$, $b \mapsto (b, 0)$.

Let $\text{Der}(Y/X, \mathcal{F})$ denote the space of derivations in \mathcal{F} .

⁴Here we say that S is of *characteristic zero* if it admits a morphism $S \rightarrow \text{Spec}(H\mathbf{Q})$, or equivalently, if the classical scheme S_{cl} is of characteristic zero.

2.8.3. Let \mathcal{F} be a 0-connected quasi-coherent module on Y .

Any derivation d of Y/X valued in \mathcal{F} gives rise to an *infinitesimal extension* $i : Y \hookrightarrow Y_d$. This is the closed immersion (in fact, nil-immersion⁵) defined as the cobase change of the trivial derivation along d , so that there is a cocartesian square

$$(2.3) \quad \begin{array}{ccc} Y_{\mathcal{F}} & \xrightarrow{d_{\text{triv}}} & Y \\ \downarrow d & & \downarrow \\ Y & \xrightarrow{i} & Y_d \end{array}$$

in the category of affine spectral schemes.

2.8.4. The following important fact often allows argument by induction along infinitesimal extensions.

Proposition 2.8.5. *Let $S = \text{Spec}(A)$ be an affine spectral scheme. Then there exists a sequence of nil-immersions of affine spectral schemes*

$$(2.4) \quad S_{\text{cl}} = S_{\leq 0} \hookrightarrow S_{\leq 1} \hookrightarrow \cdots \hookrightarrow S_{\leq n} \hookrightarrow \cdots \hookrightarrow S,$$

with $S_{\leq n} = \text{Spec}(A_{\leq n})$, satisfying the following properties:

(i) For each $n \geq 0$, the homomorphism $A \rightarrow A_{\leq n}$ identifies $A_{\leq n}$ as the n -truncation of the connective \mathcal{E}_{∞} -ring spectrum A .

(ii) The sequence is functorial in A .

(iii) The canonical morphism $A \rightarrow \varprojlim_{n \geq 0} A_{\leq n}$ is invertible.

(iv) Each morphism $S_{\leq n} \hookrightarrow S_{\leq n+1}$ ($n \geq 0$) is an infinitesimal extension by a derivation valued in $\pi_n(\mathcal{O}_S)[n+1]$.

Further, this sequence is uniquely characterized, up to isomorphism of diagrams indexed on the poset of nonnegative integers, by the property (i).

The sequence (2.4) is called the *Postnikov tower* of A .

We refer to [Lur16a, Prop. 7.1.3.19] for the proof.

2.9. The cotangent sheaf.

2.9.1. Let $p : Y \rightarrow X$ be a morphism of affine spectral schemes. We have:

Proposition 2.9.2. *The functor $\mathcal{F} \mapsto \text{Der}(Y/X, \mathcal{F})$ is representable by a connective quasi-coherent module $\mathcal{T}_{Y/X}^*$ on Y .*

The quasi-coherent module $\mathcal{T}_{Y/X}^*$ on Y is called the (relative) *cotangent sheaf* of the morphism $p : Y \rightarrow X$. We obtain the absolute cotangent sheaf \mathcal{T}_S^* by taking the relative cotangent sheaf of the unique morphism $S \rightarrow \text{Spec}(\mathbf{S})$.

2.9.3. The following fact is crucial.

Lemma 2.9.4. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be a sequence of morphisms of affine spectral schemes. Then there is a canonical exact triangle*

$$(2.5) \quad g^*(\mathcal{T}_{Y/X}^*) \rightarrow \mathcal{T}_{Z/X}^* \rightarrow \mathcal{T}_{Z/Y}^*$$

of quasi-coherent sheaves on Z .

⁵See (2.5.3).

In particular, we see that the relative cotangent sheaf $\mathcal{T}_{Y/X}^*$ is the cofibre of the canonical morphism $f^*(\mathcal{T}_X^*) \rightarrow \mathcal{T}_Y^*$.

2.9.5. The cotangent sheaf of a vector bundle has a particularly simple description:

Lemma 2.9.6. *Let S be an affine spectral scheme. For any connective quasi-coherent module \mathcal{F} on S , we have a canonical isomorphism*

$$\mathcal{T}_{S\{\mathcal{F}\}/S}^* \xrightarrow{\sim} p^*(\mathcal{F}^\vee),$$

where p denotes the structural morphism of the spectral S -scheme $S\{\mathcal{F}\}$.

2.9.7. Let $p : Y \rightarrow X$ be a morphism of spectral schemes. For any connective \mathcal{E}_∞ -ring spectrum A , A -point $y : \mathrm{Spec}(A) \rightarrow Y$, and 0-connected quasi-coherent module \mathcal{F} on Y , a *derivation at y of p with values in \mathcal{F}* is a commutative triangle

$$\begin{array}{ccc} & \mathrm{Spec}(A) & \\ \swarrow & & \searrow y \\ \mathrm{Spec}(A)_{\mathcal{F}} & \xrightarrow{d} & Y \end{array}$$

in the category of spectral X -schemes.

We write $\mathrm{Der}_y(Y/X, \mathcal{F})$ for the space of derivations at y .

2.9.8. The functor $\mathcal{F} \mapsto \mathrm{Der}_y(Y/X, \mathcal{F})$ is represented by a connective quasi-coherent module $\mathcal{T}_{Y/X,y}^*$ on $\mathrm{Spec}(A)$, called the relative cotangent sheaf of p at y :

$$\mathrm{Der}_y(Y/X, \mathcal{F}) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Qcoh}(\mathrm{Spec}(A))}(\mathcal{T}_{Y/X,y}^*, \mathcal{F}).$$

2.9.9. Given a connective \mathcal{E}_∞ -ring spectrum A and an A -point $y : \mathrm{Spec}(A) \rightarrow Y$, a connective \mathcal{E}_∞ -ring spectrum B and a B -point $y' : \mathrm{Spec}(B) \rightarrow Y$, and a morphism of affine spectral schemes $f : \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ such that $y \circ f = y'$, we obtain a canonical morphism of quasi-coherent modules on $\mathrm{Spec}(B)$

$$f^*(\mathcal{T}_{Y/X,y}^*) \rightarrow \mathcal{T}_{Y/X,y'}^*$$

which is *invertible*.

Moreover, the data of the quasi-coherent modules $\mathcal{T}_{Y/X,y}^*$, as y varies over A -points of Y (with A an arbitrary connective \mathcal{E}_∞ -ring spectrum), together with the above isomorphisms, is compatible in a homotopy coherent way, and can therefore be refined to a connective quasi-coherent module $\mathcal{T}_{Y/X}^*$ defined on the spectral scheme Y .

2.10. Smooth and étale morphisms. In this paragraph we review some standard material on smooth and étale morphisms in spectral algebraic geometry from [TV08], [Lur16a], and [Lur16b].

2.10.1. Let $p : Y \rightarrow X$ be a morphism of affine spectral schemes, with $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. We will write $\mathcal{T}_{Y/X}^*$ for the relative cotangent sheaf, i.e. the connective quasi-coherent \mathcal{O}_Y -module corresponding to the connective B -module $\mathbf{L}_{B/A}$ (= the cotangent complex).

We define:

Definition 2.10.2. (i) *The morphism p is étale if it is locally of finite presentation and the cotangent sheaf $\mathcal{T}_{Y/X}^*$ is zero.*

(ii) *The morphism p is smooth if it is locally of finite presentation and the cotangent sheaf $\mathcal{T}_{Y/X}^*$ is locally free of finite rank.*

Remark 2.10.3. Our use of the term *smooth* corresponds to *differentially smooth* in the sense of [Lur16b, Def. 11.2.2.2].

We have:

Lemma 2.10.4. (i) *The set of morphisms étale (resp. smooth) morphisms is closed under composition and base change.*

(ii) *Open immersions are étale, and étale morphisms are smooth.*

2.10.5. We now extend the above definitions to morphisms of spectral schemes, by defining them Zariski-locally on the source. That is:

Definition 2.10.6. *A morphism of spectral schemes $p : Y \rightarrow X$ is smooth (resp. étale, flat, locally of finite presentation) if there exist affine Zariski covers $(Y_\alpha \hookrightarrow Y)_\alpha$ and $(X_\beta \hookrightarrow X)_\beta$ together with the data of, for each α , an index β and a morphism of affine spectral schemes $Y_\alpha \rightarrow X_\beta$ which is smooth (resp. étale, flat, locally of finite presentation) and fits in a commutative square*

$$\begin{array}{ccc} Y_\alpha & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

Example 2.10.7. For any vector bundle E over a spectral scheme S , the projection $\pi : E \rightarrow S$ is smooth. In particular, the affine space $S\{t_1, \dots, t_n\}$ is smooth over S for each $n \geq 0$.

2.10.8. The following is a brave new version of [Gro67, Thm. 17.11.4].

Proposition 2.10.9. *A morphism $p : Y \rightarrow X$ is smooth if and only if, Zariski-locally on Y , there exists a factorization of p as a composite*

$$(2.6) \quad Y \xrightarrow{q} X\{t_1, \dots, t_n\} \xrightarrow{r} X$$

for some integer $n \geq 0$, where q is étale and r is the canonical projection.

Proof. This follows from [Lur16a, Prop. 11.2.2.1]. □

2.11. Deformation along infinitesimal extensions.

2.11.1. Let S be an affine spectral scheme and S' the infinitesimal extension of S by a derivation $d : \mathcal{T}_S^* \rightarrow \mathcal{F}$, for some 0-connected quasi-coherent module \mathcal{F} . Let X be an affine spectral scheme over S with structural morphism p .

Definition 2.11.2. *A deformation of X along the infinitesimal extension $S \hookrightarrow S'$ is an affine spectral scheme X' over S' together with an isomorphism $X \rightarrow X' \times_{S'} S$.*

In other words, a deformation of X is a cartesian square:

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ \downarrow p & & \downarrow \\ S & \hookrightarrow & S' \end{array}$$

2.11.3. From [Lur16a, Prop. 7.4.2.5], we have:

Lemma 2.11.4. *The datum of a deformation of X along $S \hookrightarrow S'$ is equivalent to the datum of a null-homotopy of the composite*

$$\mathcal{T}_{X/S}^*[-1] \rightarrow p^*(\mathcal{T}_S^*) \rightarrow p^*(\mathcal{F}).$$

Given such a null-homotopy, one obtains a derivation $d' : \mathcal{T}_X^* \rightarrow p^*(\mathcal{F})$; the deformation X' is constructed as the infinitesimal extension of X along d' .

2.11.5. For example, if p is *smooth*, then any morphism $\mathcal{T}_{X/S}^*[-1] \rightarrow p^*(\mathcal{F})$ must be null-homotopic; hence X admits a deformation along any infinitesimal extension $S \hookrightarrow S'$. If p is further *étale*, then this deformation is unique.

2.12. Push-outs of closed immersions.

2.12.1. Let $i_1 : Y \hookrightarrow X_1$ and $i_2 : Y \hookrightarrow X_2$ be closed immersions of spectral schemes.

The following is a straightforward variation on [Lur16b, Thm. 16.1.0.1] (see [Lur16b, Rem. 16.1.0.2]):

Lemma 2.12.2. *(i) There a cocartesian square of spectral schemes*

$$\begin{array}{ccc} Y & \xrightarrow{i_1} & X_1 \\ \downarrow i_2 & & \downarrow k_2 \\ X_2 & \xrightarrow{k_1} & X, \end{array}$$

where k_1 and k_2 are closed immersions.

(ii) If i_1 (resp. i_2) is a *nil-immersion*, then k_1 (resp. k_2) is a *nil-immersion*.

2.13. Lifting smooth morphisms along closed immersions.

2.13.1. The following is a spectral version of [Gro67, Prop. 18.1.1]:

Proposition 2.13.2. *Let $i : Z \hookrightarrow S$ be a closed immersion of spectral schemes. For any smooth (resp. étale) morphism $p : X \rightarrow Z$, there exists, Zariski-locally on X , a smooth (resp. étale) morphism $q : Y \rightarrow S$, and a cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow p & & \downarrow q \\ Z & \longrightarrow & S. \end{array}$$

Proof. First we consider the étale case. The question being Zariski-local, we may assume that S , Z and X are affine. Consider the Postnikov towers (Proposition 2.8.5)

$$S_{\text{cl}} = S_{\leq 0} \hookrightarrow S_{\leq 1} \hookrightarrow \cdots \hookrightarrow S_{\leq n} \hookrightarrow \cdots \hookrightarrow S$$

$$Z_{\text{cl}} = Z_{\leq 0} \hookrightarrow Z_{\leq 1} \hookrightarrow \cdots \hookrightarrow Z_{\leq n} \hookrightarrow \cdots \hookrightarrow Z$$

for S and Z , respectively. Since p is flat, the Postnikov tower for X is identified with the base change of the Postnikov tower of Z .

For a fixed integer $n \geq 0$, consider the following claim:

(*) There exists, Zariski-locally on $X_{\leq n}$, an étale morphism $q_{\leq n} : Y_{\leq n} \rightarrow S_{\leq n}$ and a cartesian square

$$\begin{array}{ccc} X_{\leq n} & \hookrightarrow & Y_{\leq n} \\ \downarrow p_{\leq n} & & \downarrow q_{\leq n} \\ Z_{\leq n} & \hookrightarrow & S_{\leq n}. \end{array}$$

Note that it suffices to show that (*) holds for each $n \geq 0$, since we can conclude by passing to filtered colimits. For $n = 0$, the claim is [Gro67, Prop. 18.1.1].

We proceed by induction; assume that the claim holds for a fixed n . We define $Y_{\leq n+1}$ to be the deformation of $Y_{\leq n}$ along the infinitesimal extension $S_{\leq n} \hookrightarrow S_{\leq n+1}$, which exists by Lemma 2.11.4. Note that $X_{\leq n+1}$ is itself a deformation of $X_{\leq n}$ along the infinitesimal extension $Z_{\leq n} \hookrightarrow Z_{\leq n+1}$. That the resulting square is cartesian is a straightforward verification.

In the smooth case, the claim follows from the étale case and from Proposition 2.10.9. \square

3. THE BRAVE NEW MOTIVIC HOMOTOPY CATEGORY

In this section, we construct the brave new motivic homotopy category over any spectral base scheme S . We adopt the following notation:

By Sm_S we denote the (essentially small) category of smooth spectral schemes of finite type over S . In what follows, we will always say “smooth” to mean “smooth of finite type”.

By *fibred space* or simply *space* over S , we will mean a presheaf on Sm_S . We write $\mathrm{Spc}(S)$ for the category of fibred spaces over S , and h_S for the Yoneda embedding $\mathrm{Sm}_S \hookrightarrow \mathrm{Spc}(S)$.

3.1. Nisnevich excision.

3.1.1. Let S be a spectral scheme. A *Nisnevich square* over S is a cartesian square of spectral schemes over S

$$(3.1) \quad \begin{array}{ccc} U \times_X V & \xhookrightarrow{k} & V \\ \downarrow q & & \downarrow p \\ U & \xhookrightarrow{j} & X \end{array}$$

such that j is an open immersion, p is étale, and there exists a closed immersion $Z \hookrightarrow X$ complementary to j such that the induced morphism $p^{-1}(Z) \rightarrow Z$ is invertible.

3.1.2. We let $\mathbb{E}_S^{\mathrm{Nis}}$ denote the set of Nisnevich squares in the category Sm_S . It is clear that this defines an excision structure in the sense of Definition A.1.2.

The following technical lemma will be useful:

Lemma 3.1.3. *The excision structure $\mathbb{E}_S^{\mathrm{Nis}}$ is topological in the sense of Definition A.2.2.*

Proof. The axiom (EXC2) is clear, since every open immersion of spectral schemes is a monomorphism.

For (EXC3), let Q be a Nisnevich square of the form (3.1) and consider the induced cartesian square

$$\begin{array}{ccc} W & \xrightarrow{k} & V \\ \downarrow \Delta_{W/U} & & \downarrow \Delta_{V/X} \\ W \times_U W & \xrightarrow{(k, k)} & V \times_X V. \end{array}$$

It is clear that the lower horizontal morphism is an open immersion. The morphism $\Delta_{V/X}$ is étale (in fact an open immersion) because p is étale, and it is clear that it induces an isomorphism on the complementary reduced closed subscheme. \square

3.1.4. Let $\mathrm{Spc}_{\mathrm{Nis}}(S)$ denote the full subcategory of $\mathrm{Spc}(S)$ spanned by fibred spaces \mathcal{F} satisfying *Nisnevich excision*, i.e.:

- (1) The space $\mathcal{F}(\emptyset)$ is contractible, where \emptyset is the empty spectral scheme.
- (2) For every Nisnevich square of the form (3.1), the induced commutative square of spaces

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(U \times_X V, \mathcal{F}) \end{array}$$

is cartesian.

We have:

Proposition 3.1.5. (i) *The category $\mathrm{Spc}_{\mathrm{Nis}}(S)$ is an accessible left localization of the category $\mathrm{Spc}(S)$, i.e. there is an accessible localization functor $L_{\mathrm{Nis}} : \mathrm{Spc}(S) \rightarrow \mathrm{Spc}_{\mathrm{Nis}}(S)$, left adjoint to the inclusion. In particular, $\mathrm{Spc}_{\mathrm{Nis}}(S)$ is an arena.*

(ii) *The localization functor $\mathcal{F} \mapsto L_{\mathrm{Nis}}(\mathcal{F})$ is left-exact, i.e. commutes with finite limits. In particular, $\mathrm{Spc}_{\mathrm{Nis}}(S)$ is a topos, and has universality of colimits.*

(iii) *The full subcategory $\mathrm{Spc}_{\mathrm{Nis}}(S) \subset \mathrm{Spc}(S)$ is stable under filtered colimits.*

Proof. Claims (i) and (ii) follow from Corollary A.2.10, in view of Lemma 3.1.3.

Claim (iii) follows from Lemma A.1.6. \square

We will say that a morphism of fibred spaces is a *Nisnevich-local equivalence* if it becomes invertible after applying the localization functor L_{Nis} .

Remark 3.1.6. Theorem A.2.9 implies that the property of Nisnevich excision is equivalent to Čech descent with respect to the Grothendieck topology generated by Nisnevich squares. It follows from [Lur16b, Thm. 3.7.5.1] that, over quasi-compact quasi-separated spectral schemes, this Grothendieck topology coincides with the Nisnevich topology as constructed by Lurie in [Lur16b, §3.7].

In view of this, we will use the terms *Nisnevich sheaf* and *Nisnevich-excise presheaf* interchangeably.

3.1.7. The following lemma follows from [Lur09a, Prop. 5.5.8.10, (3)]:

Lemma 3.1.8. *Let $(X_\alpha)_\alpha$ be a finite family of smooth spectral schemes over S . Then the canonical morphism of presheaves*

$$\bigsqcup_\alpha h_S(X_\alpha) \rightarrow h_S(\bigsqcup_\alpha X_\alpha)$$

is a Nisnevich-local equivalence.

By [Lur09a, Lem. 5.5.8.14] it follows that the category of Nisnevich sheaves is generated under sifted colimits by the representables. In fact, we can say even more:

Lemma 3.1.9. *The category of Nisnevich sheaves is generated under sifted colimits by the representable presheaves $h_S(X)$, where $X = \operatorname{Spec}(A)$ is an affine spectral scheme which is smooth over S .*

Proof. Let $h_S(X)$ be a representable presheaf over S . Since $h_S(X)$ satisfies Nisnevich descent, we can assume X is separated over S , by choosing an affine Zariski cover of X where the pairwise intersections are separated. Then we repeat the same argument to assume X is affine, by choosing an affine cover where the pairwise intersections are affine. \square

3.2. Homotopy invariance.

3.2.1. Let \mathbf{I} denote the spectral affine line over the sphere spectrum, i.e. $\mathbf{I} = \operatorname{Spec}(\mathbf{S})\{t\}$ in the notation of (2.6.9).

We say that a fibred space \mathcal{F} over S satisfies *\mathbf{I} -homotopy invariance* if for every smooth spectral scheme X over S , the morphism of spaces

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X \times \mathbf{I}, \mathcal{F}),$$

induced by the projection $X \times \mathbf{I} \rightarrow X$, is invertible.

Note that $X \times \mathbf{I}$ is just another way to write $X\{t\}$, in the notation of (2.6.9).

3.2.2. Let $\operatorname{Spc}_{\mathbf{I}}(S)$ denote the full subcategory of $\operatorname{Spc}(S)$ spanned by \mathbf{I} -homotopy invariant spaces.

We have:

Proposition 3.2.3. *(i) The category $\operatorname{Spc}_{\mathbf{I}}(S)$ is an accessible left localization of the category $\operatorname{Spc}(S)$, i.e. there is an accessible localization functor $L_{\mathbf{I}} : \operatorname{Spc}(S) \rightarrow \operatorname{Spc}_{\mathbf{I}}(S)$, left adjoint to the inclusion. In particular, $\operatorname{Spc}_{\mathbf{I}}(S)$ is an arena.*

(ii) For every fibred space \mathcal{F} , there is a canonical isomorphism

$$(3.2) \quad \Gamma(X, L_{\mathbf{I}}(\mathcal{F})) = \varinjlim_{(Y \rightarrow X) \in (\mathbf{A}_X)^{\operatorname{op}}} \Gamma(Y, \mathcal{F})$$

for each smooth spectral scheme X over S . Here $(\mathbf{A}_X)^{\operatorname{op}}$ is a sifted small category, opposite to the full subcategory of $\operatorname{Sm}/_X$ spanned by compositions of \mathbf{I} -projections.

(iii) The localization functor $\mathcal{F} \mapsto L_{\mathbf{I}}(\mathcal{F})$ commutes with finite products.

(iv) The category $\operatorname{Spc}_{\mathbf{I}}(S)$ has universality of colimits.

(v) The full subcategory $\operatorname{Spc}_{\mathbf{I}}(S) \subset \operatorname{Spc}(S)$ is stable under colimits.

Proof. Let \mathbf{A} denote the set of projections $X \times \mathbf{I} \rightarrow X$ for each smooth spectral scheme X over S ; note that the condition of \mathbf{I} -homotopy invariance is nothing else than \mathbf{A} -invariance in the sense of Definition A.3.2.

Hence the claims follow from Lemma A.3.3 and Proposition A.3.6. \square

We will say that a morphism of fibred spaces is an \mathbf{I} -homotopy equivalence if it becomes invertible after applying the \mathbf{I} -localization functor $L_{\mathbf{I}}$.

3.2.4. Let $p : \mathbf{I} \rightarrow \mathrm{Spec}(\mathbf{S})$ denote the projection to the terminal spectral scheme. This admits two sections i_0 and i_1 , the zero and unit sections, which are closed immersions of spectral schemes. They correspond to the morphisms $\mathbf{S}\{t\} \rightarrow \mathbf{S}$ sending t to 0 and 1, respectively (which are defined, as usual, up to a contractible space of choices).

Remark 3.2.5. Following [MV99, §2.3], it is possible to use the sections i_0 and i_1 to construct a cosimplicial diagram of spectral schemes given degree-wise by

$$\Delta_{\mathbf{S}}^p = \mathbf{S} \times \mathbf{I}^p = \mathbf{S} \times \mathrm{Spec}(\mathbf{S}\{t_1, \dots, t_p\}).$$

for each $[p] \in \mathbf{\Delta}$.

The localization functor $L_{\mathbf{I}}$ can then be computed by the formula

$$(3.3) \quad \Gamma(X, L_{\mathbf{I}}(\mathcal{F})) = \varinjlim_{[p] \in \mathbf{\Delta}^{\mathrm{op}}} \Gamma(X \times \Delta_{\mathbf{S}}^p, \mathcal{F}).$$

For our purposes, the formula (3.2) will suffice; both indexing categories $(\mathbf{A}_{\mathbf{X}})^{\mathrm{op}}$ and $\mathbf{\Delta}^{\mathrm{op}}$ are *sifted*, which is the only property we need.

3.2.6. Given two morphisms $f, g : \mathcal{F} \rightrightarrows \mathcal{G}$ of presheaves on Sm/S , an *elementary \mathbf{I} -homotopy* from f to g is a morphism

$$h_S(\mathbf{S} \times \mathbf{I}) \times \mathcal{F} \rightarrow \mathcal{G}$$

whose restriction to $\mathcal{F} = h_S(\mathbf{S}) \times \mathcal{F}$ along i_0 (resp. i_1) is isomorphic to f (resp. g).

We say that f and g are *\mathbf{I} -homotopic* if there exists a sequence of elementary \mathbf{I} -homotopies connecting them. In this case the induced morphisms $L_{\mathbf{I}}(\mathcal{F}) \rightrightarrows L_{\mathbf{I}}(\mathcal{G})$ coincide.

3.2.7. A morphism of presheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is called a *strict \mathbf{I} -homotopy equivalence* if there exists a morphism $\psi : \mathcal{G} \rightarrow \mathcal{F}$ such that the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are \mathbf{I} -homotopic to the identities.

Note that any strict \mathbf{I} -homotopy equivalence is an \mathbf{I} -homotopy equivalence.

3.3. Motivic spaces.

3.3.1. We define:

Definition 3.3.2. A motivic space over S is a fibred space \mathcal{F} satisfying Nisnevich excision and \mathbf{I} -homotopy invariance.

Let $\mathrm{MotSpc}^{\varepsilon\infty}(S)$ denote the full subcategory of $\mathrm{Spc}(S)$ spanned by motivic spaces.

3.3.3. For any smooth spectral scheme X over S , we let $M_S(X) := L_{\mathrm{mot}}(h_S(X))$. We have canonical bifunctorial isomorphisms

$$\mathrm{Maps}_{\mathrm{MotSpc}^{\varepsilon\infty}(S)}(M_S(X), \mathcal{F}) = \Gamma(X, \mathcal{F})$$

for every motivic space \mathcal{F} over S .

3.3.4. We have:

Proposition 3.3.5. (i) The category $\mathrm{MotSpc}^{\varepsilon\infty}(S)$ is an accessible left localization of $\mathrm{Spc}(S)$, i.e. there is an accessible localization functor $L_{\mathrm{mot}} : \mathrm{Spc}(S) \rightarrow \mathrm{MotSpc}^{\varepsilon\infty}(S)$, left adjoint to the inclusion. In particular, $\mathrm{MotSpc}^{\varepsilon\infty}(S)$ is an arena.

(ii) The arena $\mathrm{MotSpc}^{\varepsilon\infty}(S)$ admits a cartesian monoidal structure, and the localization functor $\mathcal{F} \mapsto L_{\mathrm{mot}}(\mathcal{F})$ lifts to a symmetric monoidal functor.

(iii) The localization functor $\mathcal{F} \mapsto L_{\text{mot}}(\mathcal{F})$ can be described as the transfinite composite

$$(3.4) \quad L_{\text{mot}}(\mathcal{F}) = \varinjlim_{n \geq 0} (L_{\mathbf{I}} \circ L_{\text{Nis}})^{\circ n}(\mathcal{F}).$$

(iv) The full subcategory $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S}) \subset \text{Spc}(\mathcal{S})$ is stable under filtered colimits.

(v) The category $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})$ has the property of universality of colimits.

(vi) The category $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})$ is generated under sifted colimits by presheaves of the form $M_S(X)$, where $X = \text{Spec}(A)$ is an affine spectral scheme which is smooth over S .

Proof. Note that the category $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})$ is nothing else than the category $\mathbf{H}_{\mathbf{E}_S^{\text{Nis}}, \mathbf{A}_S}(\text{Sm}/S)$, the unstable homotopy theory associated to the Nisnevich excision structure $\mathbf{E}_S^{\text{Nis}}$ and the set of \mathbf{I} -projections $X \times \mathbf{I} \rightarrow X$ (for X a smooth spectral S -scheme). Hence claims (i)–(v) follow from the general results collected in Sect. A.

Claim (vi) follows directly from Lemma 3.1.9. \square

We will say that a morphism of fibred spaces is a *motivic equivalence* if it induces an isomorphism after applying the motivic localization functor L_{mot} .

3.4. Pointed motivic spaces.

3.4.1. Let $\text{Spc}(\mathcal{S})_\bullet$ denote the arena of pointed objects in $\text{Spc}(\mathcal{S})$, i.e. pairs (\mathcal{F}, x) with \mathcal{F} a fibred space over S , and $x : \text{pt}_S \rightarrow \mathcal{F}$ a morphism from the terminal space.

The forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$ admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}_+$, which freely adjoins a point.

3.4.2. Let $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})_\bullet$ denote the arena of pointed objects in $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})$.

We have:

Proposition 3.4.3. (i) The category $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})_\bullet$ is an accessible left localization of $\text{Spc}(\mathcal{S})_\bullet$, i.e. there is an accessible localization functor $L_{\text{mot}} : \text{Spc}(\mathcal{S})_\bullet \rightarrow \text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})_\bullet$, left adjoint to the inclusion.

(ii) The arena $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})_\bullet$ admits a symmetric monoidal structure. The functors $\mathcal{F} \mapsto \mathcal{F}_+$ and $\mathcal{F} \mapsto L_{\text{mot}}(\mathcal{F})$ lift to symmetric monoidal functors.

(iii) The localization functor $\mathcal{F} \mapsto L_{\text{mot}}(\mathcal{F})$ satisfies the formula

$$(3.5) \quad L_{\text{mot}}(\mathcal{F}_+) = L_{\text{mot}}(\mathcal{F})_+$$

for every space \mathcal{F} over S .

(iv) The category $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})_\bullet$ is generated under sifted colimits by objects of the form $M_S(X)_+$, where $X = \text{Spec}(A)$ is an affine spectral scheme which is smooth over S .

Proof. The claims (i)–(iii) follow from the general statements of Paragraph A.5.

Claim (iv) follows from point (vi) of Proposition 3.3.5 and Lemma A.5.12. \square

We will write \wedge_S for the monoidal product on $\text{MotSpc}^{\varepsilon_\infty}(\mathcal{S})_\bullet$.

3.5. Thom spaces.

3.5.1. Let E be a vector bundle over S . Let $j : E^\times \hookrightarrow E$ denote the open immersion complementary to the zero section.

Definition 3.5.2. *The Thom space $\mathrm{Th}_S(E)$ is the pointed presheaf defined as the cofibre*

$$\mathrm{Th}_S(E) = \mathrm{Cofib}(h_S(E^\times)_+ \rightarrow h_S(E)_+).$$

We let $\mathbf{T}_S = \mathrm{Th}_S(S \times \mathbf{I})$ denote the Thom space of the spectral affine line.

3.5.3. Note that the projection $S \times \mathbf{I} \rightarrow S$ induces a canonical morphism

$$\mathrm{Cofib}(h_S(S \times \mathbf{I}^\times)_+ \rightarrow h_S(S \times \mathbf{I})_+) \rightarrow \mathrm{Cofib}(h_S(S \times \mathbf{I}^\times)_+ \rightarrow (\mathrm{pt}_S)_+).$$

That is, we get a canonical morphism of pointed spaces over S

$$(3.6) \quad \mathbf{T}_S \rightarrow \Sigma_{\mathbf{S}^1}(h_S(S \times \mathbf{I}^\times)),$$

where we view $h_S(S \times \mathbf{I}^\times)$ as pointed at the unit section $i_1 : S \hookrightarrow S \times \mathbf{I}^\times$, say.

This is obviously an \mathbf{I} -homotopy equivalence:

Lemma 3.5.4. *There is a canonical isomorphism*

$$L_{\mathrm{mot}}(\mathbf{T}_S) = \Sigma_{\mathbf{S}^1}(M_S(S \times \mathbf{I}^\times))$$

of pointed motivic spaces over S .

3.5.5. Consider the endomorphism $S\{t\} \rightarrow S\{t\}$ induced from the morphism of \mathcal{E}_∞ -ring spectra $\mathbf{S}\{t\} \rightarrow \mathbf{S}\{t\}$ sending $t \mapsto -t$. It induces an endomorphism

$$\langle -1 \rangle : \mathbf{T}_S \rightarrow \mathbf{T}_S$$

of the Thom space.

We have:

Lemma 3.5.6. *The morphism $\sigma : \mathbf{T}_S \otimes \mathbf{T}_S \rightarrow \mathbf{T}_S \otimes \mathbf{T}_S$, permuting the two factors, is \mathbf{I} -homotopic to the morphism $\langle -1 \rangle \otimes \mathrm{id}_{\mathbf{T}_S}$.*

This implies directly:

Corollary 3.5.7. *The pointed presheaf $L_{\mathrm{mot}}(\mathbf{T}_S)$ is 3-symmetric with respect to the smash product \wedge_S .*

3.6. Motivic spectra.

3.6.1. Let $\mathrm{Spt}(S)$ denote the arena $\mathrm{Spt}_{\mathbf{T}_S}(\mathrm{Spc}(S)_\bullet)$ of \mathbf{T}_S -spectrum objects in the arena of pointed fibred spaces over S .

This is formed by formally adjoining a monoidal inverse to the object \mathbf{T}_S in the symmetric monoidal arena $\mathrm{Spc}(S)_\bullet$; see Paragraph A.6.

We call objects of $\mathrm{Spt}(S)$ (*fibred*) *spectra* over S . These are sequences $(\mathcal{F}_n)_{n \geq 0}$ of pointed fibred spaces \mathcal{F}_n , together with isomorphisms

$$\mathcal{F}_n \xrightarrow{\sim} \Omega_{\mathbf{T}}(\mathcal{F}_{n+1}).$$

3.6.2. Similarly, let $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$ denote the arena $\text{Spt}_{\text{L}_{\text{mot}}(\mathbf{T}_{\mathcal{S}})}(\text{MotSpc}^{\mathcal{E}\infty}(\mathcal{S})_{\bullet})$.

There is a canonical adjunction

$$(3.7) \quad \Sigma_{\mathbf{T}}^{\infty} : \text{MotSpc}^{\mathcal{E}\infty}(\mathcal{S})_{\bullet} \rightleftarrows \text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S}) : \Omega_{\mathbf{T}}^{\infty}.$$

We have:

Proposition 3.6.3. (i) *The category $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$ is an accessible left localization of $\text{Spt}(\mathcal{S})$, i.e. there is an accessible localization functor $\text{L}_{\text{mot}} : \text{Spt}(\mathcal{S}) \rightarrow \text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$, left adjoint to the inclusion.*

(ii) *The arena $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$ is symmetric monoidal, and the functors $\Sigma_{\mathbf{T}}^{\infty}$ and L_{mot} lift to symmetric monoidal functors. The object $\Sigma_{\mathbf{T}}^{\infty}(\mathbf{T})$ is monoidally invertible.*

(iii) *The arena $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$ is stable, and the localization functor $\mathcal{F} \mapsto \text{L}_{\text{mot}}(\mathcal{F})$ is exact.*

(iv) *The localization functor $\mathcal{F} \mapsto \text{L}_{\text{mot}}(\mathcal{F})$ satisfies the formula*

$$\text{L}_{\text{mot}}(\Sigma_{\mathbf{T}}^{\infty-k}(\text{h}_{\mathcal{S}}(\mathcal{X})_{+})) = \Sigma_{\mathbf{T}}^{\infty-k}(\text{M}_{\mathcal{S}}(\mathcal{X})_{+})$$

for each smooth spectral scheme \mathcal{X} over \mathcal{S} .

(v) *The localization functor $\mathcal{F} \mapsto \text{L}_{\text{mot}}(\mathcal{F})$ can be computed as the composite*

$$(3.8) \quad \text{L}_{\text{mot}} = \text{L}_{\mathbf{I}} \circ \text{L}_{\text{Nis}}.$$

(vi) *The category $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$ is generated under sifted colimits by objects of the form $\Sigma_{\mathbf{T}}^{\infty-n}(\text{M}_{\mathcal{S}}(\mathcal{X})_{+})$, where $\mathcal{X} = \text{Spec}(\mathcal{A})$ is an affine spectral scheme which is smooth over \mathcal{S} , and $n \geq 0$.*

Proof. The claims follow from the general results of Paragraph A.6, together with Lemma 3.5.4, Corollary 3.5.7, and point (iv) of Proposition 3.4.3. \square

We will write $\otimes_{\mathcal{S}}$ for the monoidal product on $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$.

We will abuse notation by writing \mathbf{T} also for the monoidally invertible object $\Sigma_{\mathbf{T}}^{\infty}(\mathbf{T})$. Let $\mathbf{T}^{\otimes 0} = \mathbf{1}_{\mathcal{S}}$ be the monoidal unit, and let $\mathbf{T}^{\otimes(-1)}$ be a monoidal inverse to \mathbf{T} ; for $n > 1$, write $\mathbf{T}^{\otimes(-n)} = (\mathbf{T}^{\otimes(-1)})^{\otimes n}$.

3.7. Comparison with classical motivic homotopy theory.

3.7.1. Given a classical scheme \mathcal{S} , let $\text{MotSpc}^{\text{cl}}(\mathcal{S})$ (resp. $\text{MotSpt}^{\text{cl}}(\mathcal{S})$) denote the classical unstable (resp. stable) motivic homotopy category over \mathcal{S} . See e.g. [Hoy15a, Appendix C] for an $(\infty, 1)$ -categorical construction in this generality.

We have:

Theorem 3.7.2. *Let \mathcal{S} be a classical scheme of characteristic zero. Then there are canonical equivalences of categories*

$$\begin{aligned} \text{MotSpc}^{\mathcal{E}\infty}(\mathcal{S}) &= \text{MotSpc}^{\text{cl}}(\mathcal{S}), \\ \text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S}) &= \text{MotSpt}^{\text{cl}}(\mathcal{S}). \end{aligned}$$

Proof. From [Lur16b, Props. 11.2.4.1 and 11.2.4.4] it follows that a morphism of spectral schemes $p : \mathcal{X} \rightarrow \mathcal{S}$ is smooth if and only if \mathcal{X} is classical and p is smooth in the classical sense. According to Proposition 2.7.4 we know that the brave new affine line $\mathcal{S}\{t\}$ coincides with the classical affine line $\mathcal{S}[t]$. This also implies that the Thom space $\mathbf{T}_{\mathcal{S}}$ coincides with the Thom space of the classical affine line. All this together shows that our categories $\text{MotSpc}^{\mathcal{E}\infty}(\mathcal{S})$ and $\text{MotSpt}^{\mathcal{E}\infty}(\mathcal{S})$ coincide with the categories $\text{MotSpc}^{\text{cl}}(\mathcal{S})$ and $\text{MotSpt}^{\text{cl}}(\mathcal{S})$ by construction. \square

4. INVERSE AND DIRECT IMAGE FUNCTORIALITY

4.1. For motivic spaces.

4.1.1. Let $f : T \rightarrow S$ be a morphism of spectral schemes. The direct image functor

$$(4.1) \quad f_*^{\text{Spc}} : \text{Spc}(T) \rightarrow \text{Spc}(S)$$

is defined as restriction along the base change functor $\text{Sm}_S \rightarrow \text{Sm}_T$.

According to Lemma A.4.5, its left adjoint f_{Spc}^* , the inverse image functor, is uniquely characterized by commutativity with colimits and the formula

$$(4.2) \quad f_{\text{Spc}}^*(h_S(X)) = h_T(X \times_S T)$$

for smooth spectral schemes X over S .

4.1.2. Note that the base change functor $\text{Sm}_S \rightarrow \text{Sm}_T$ preserves Nisnevich covering families and **I**-projections. It follows that the inverse image functor f_{Spc}^* preserves Nisnevich-local equivalences and **I**-homotopy equivalences.

By adjunction, its right adjoint f_*^{Spc} preserves Nisnevich sheaves and **I**-homotopy invariant spaces and induces a functor $f_*^{\text{MotSpc}} : \text{MotSpc}^{\mathcal{E}^\infty}(T) \rightarrow \text{MotSpc}^{\mathcal{E}^\infty}(S)$. This admits a left adjoint f_{MotSpc}^* given by the formula

$$f_{\text{MotSpc}}^*(\mathcal{F}) = L_{\text{mot}}(f_{\text{Spc}}^*(\mathcal{F})).$$

According to Proposition A.4.7, this is characterized by commutativity with colimits and the formula

$$(4.3) \quad f_{\text{MotSpc}}^*(M_S(X)) = M_T(X \times_S T).$$

4.1.3. Both the direct and inverse image functors are symmetric monoidal:

Lemma 4.1.4. *The functor f_*^{Spc} (resp. f_*^{MotSpc}) admits a canonical symmetric monoidal structure.*

Proof. Since the respective symmetric monoidal structures are cartesian, it suffices to show that f_* commutes with finite products. In fact, it commutes with arbitrary limits since it is a right adjoint. \square

Lemma 4.1.5. *The functor f_{Spc}^* (resp. f_{MotSpc}^*) admits a canonical symmetric monoidal structure.*

Proof. By adjunction from Lemma 4.1.4, we obtain a canonical structure of colax symmetric monoidal functor on f^* . That is, there are canonical morphisms

$$(4.4) \quad f^*(\mathcal{F} \times_S \mathcal{G}) \rightarrow f^*(\mathcal{F}) \times_T f^*(\mathcal{G})$$

for any two spaces \mathcal{F} and \mathcal{G} over S . It suffices to show that these morphisms are invertible.

Since f_{Spc}^* commutes with colimits, and the cartesian product commutes with colimits in each argument, one reduces to the case of representables, in which case the claim is clear. For f_{MotSpc}^* , the claim follows from the first because motivic localization commutes with finite products. \square

4.2. For pointed motivic spaces.

4.2.1. Let $f : T \rightarrow S$ be a morphism of spectral schemes. Since f_*^{Spc} preserves the terminal object, it induces a functor $f_*^{\text{Spc}\bullet} : \text{Spc}(T)_\bullet \rightarrow \text{Spc}(S)_\bullet$ given on objects by the formula

$$f_*^{\text{Spc}\bullet}(\mathcal{G}, y) = (f_*^{\text{Spc}}(\mathcal{G}), f_*^{\text{Spc}}(y)).$$

Its left adjoint $f_{\text{Spc}\bullet}^* : \text{Spc}(T)_\bullet \rightarrow \text{Spc}(S)_\bullet$ is uniquely characterized, according to Lemma A.5.4, by the fact that it commutes with sifted colimits and with the functor $\mathcal{F} \mapsto \mathcal{F}_+$:

$$(4.5) \quad f_{\text{Spc}\bullet}^*(\mathcal{F}_+) = f_{\text{Spc}}^*(\mathcal{F})_+$$

for any space \mathcal{F} over S .

Explicitly, it is given on objects by the formula

$$f_{\text{Spc}\bullet}^*(\mathcal{F}, x) = (f_{\text{Spc}}^*(\mathcal{F}), f_{\text{Spc}}^*(x))$$

for each pointed space (\mathcal{F}, x) over S .

4.2.2. The direct image functor $f_*^{\text{Spc}\bullet}$ preserves the properties of Nisnevich descent and \mathbf{I} -homotopy invariance, and induces a functor f_*^{MotSpc} . Its left adjoint $f_{\text{MotSpc}\bullet}^*$ is given by composing $f_{\text{Spc}\bullet}^*$ with the motivic localization functor:

$$(4.6) \quad f_{\text{MotSpc}\bullet}^* := L_{\text{mot}} f_{\text{Spc}\bullet}^*.$$

It is uniquely characterized, according to Lemma A.5.12, by commutativity with sifted colimits and the formula

$$(4.7) \quad f_{\text{MotSpc}\bullet}^*(\mathcal{F}_+) = f_{\text{MotSpc}}^*(\mathcal{F})_+.$$

4.2.3.

Lemma 4.2.4. *The inverse image functor $f_{\text{Spc}\bullet}^*$ (resp. $f_{\text{MotSpc}\bullet}^*$) admits a canonical symmetric monoidal structure.*

Proof. For $f_{\text{Spc}\bullet}^*$, this follows directly from the universal property of Lemma A.5.6 and the formula (4.5). For $f_{\text{MotSpc}\bullet}^*$ it follows from the universal property of Lemma A.5.14 and the formula (4.7). \square

4.3. For motivic spectra.

4.3.1. For each spectral scheme S , recall that \mathbf{T}_S denotes the Thom space of the spectral affine line $S\{t\}$.

Lemma 4.3.2. *For any morphism of spectral schemes $f : T \rightarrow S$, there is a canonical isomorphism*

$$(4.8) \quad f_{\text{Spc}\bullet}^*(\mathbf{T}_S) = \mathbf{T}_T$$

of pointed spaces over T .

Proof. The functor $f_{\text{Spc}\bullet}^*$ preserves cofibre sequences, and we have canonical isomorphisms

$$\begin{aligned} f_{\text{Spc}\bullet}^*(h_S(S \times \mathbf{I})_+) &= h_T(T \times \mathbf{I})_+, \\ f_{\text{Spc}\bullet}^*(h_S(S \times \mathbf{I}^\times)_+) &= h_T(T \times \mathbf{I}^\times)_+ \end{aligned}$$

by the formula (4.2). \square

4.3.3. Let $f : T \rightarrow S$ be a morphism of spectral schemes. Since $f_{\mathrm{Spc}\bullet}^*$ is monoidal (Lemma 4.2.4), it commutes with \mathbf{T} -suspensions.

By construction of $\mathrm{Spt}(S)$ and Lemma 4.3.2, there exists a unique functor $f_{\mathrm{Spt}}^* : \mathrm{Spt}(S) \rightarrow \mathrm{Spt}(T)$ which commutes with colimits and with the functor $\Sigma_{\mathbf{T}}^\infty$, i.e.:

$$(4.9) \quad f_{\mathrm{Spt}}^* \Sigma_{\mathbf{T}}^\infty = \Sigma_{\mathbf{T}}^\infty f_{\mathrm{Spc}\bullet}^*.$$

Alternatively, we can use Lemma A.6.11 to describe f_{Spt}^* as the unique functor which commutes with filtered colimits and with the functor $\Sigma_{\mathbf{T}}^{\infty-n}$ for each $n \geq 0$:

$$(4.10) \quad f_{\mathrm{Spt}}^* \Sigma_{\mathbf{T}}^{\infty-n} = \Sigma_{\mathbf{T}}^{\infty-n} f_{\mathrm{Spc}\bullet}^*.$$

4.3.4. Let f_*^{Spt} be the right adjoint of f_{Spt}^* . This can be described as the unique functor which commutes with limits and with the functor Ω^∞ , i.e.:

$$(4.11) \quad \Omega^\infty f_*^{\mathrm{Spt}} = f_*^{\mathrm{Spc}\bullet} \Omega^\infty$$

It is given on objects by the assignment

$$\mathbb{E} = (\mathcal{F}_n)_n \mapsto f_*(\mathbb{E}) = (f_*^{\mathrm{Spc}\bullet}(\mathcal{F}_n))_n.$$

4.3.5. The direct image functor f_*^{Spt} preserves motivic spectra and induces a functor f_*^{MotSpt} .

We let f_{MotSpt}^* be its left adjoint, the symmetric monoidal functor $L_{\mathrm{mot}} f_{\mathrm{Spt}}^*$. This is the unique functor which commutes with colimits and with the functor $\Sigma_{\mathbf{T}}^\infty$, i.e.:

$$(4.12) \quad f_{\mathrm{MotSpt}}^* \Sigma_{\mathbf{T}}^\infty = \Sigma_{\mathbf{T}}^\infty f_{\mathrm{MotSpc}\bullet}^*.$$

4.3.6. Using the universal properties of Lemma A.6.13 and Lemma A.6.25, we get:

Lemma 4.3.7. *The functor f_{Spt}^* (resp. f_{MotSpt}^*) admits a canonical symmetric monoidal structure.*

5. FUNCTORIALITY ALONG SMOOTH MORPHISMS

As per our conventions, smooth morphisms will be assumed to be of finite type.

5.1. The functor p_{\sharp} .

5.1.1. Let $p : X \rightarrow S$ be a smooth morphism (of finite type) of spectral schemes. In this case the base change functor admits a right adjoint, the forgetful functor $\mathrm{Sm}/_X \rightarrow \mathrm{Sm}/_S$:

$$(Y \rightarrow X) \mapsto (Y \rightarrow X \xrightarrow{p} S).$$

It follows that the functor $p_{\mathrm{Spc}\bullet}^*$ coincides with restriction along the forgetful functor, and admits a left adjoint

$$p_{\sharp}^{\mathrm{Spc}} : \mathrm{Spc}(T) \rightarrow \mathrm{Spc}(S).$$

This is uniquely characterized, according to Lemma A.4.5, by commutativity with colimits and the formula

$$(5.1) \quad p_{\sharp}^{\mathrm{Spc}}(h_X(Y)) = h_S(Y).$$

for smooth spectral schemes Y over X .

5.1.2. Since the forgetful functor $\mathrm{Sm}/_X \rightarrow \mathrm{Sm}/_S$ preserves Nisnevich covering families and \mathbf{A}^1 -projections, it follows that $p_{\sharp}^{\mathrm{Spc}}$ preserves Nisnevich-local equivalences and \mathbf{A}^1 -homotopy equivalences.

In particular its right adjoint p_{Spc}^* preserves Nisnevich descent and \mathbf{A}^1 -homotopy invariance, and induces a morphism p_{MotSpc}^* on motivic spaces.

Its left adjoint $p_{\sharp}^{\mathrm{MotSpc}}$ is given by applying $p_{\sharp}^{\mathrm{Spc}}$ and then the localization functor L_{mot} :

$$p_{\sharp}^{\mathrm{MotSpc}}(\mathcal{F}) = L_{\mathrm{mot}}(p_{\sharp}^{\mathrm{Spc}}(\mathcal{F})).$$

It is uniquely characterized, according to Proposition A.4.7, by commutativity with colimits and the formula

$$(5.2) \quad p_{\sharp}^{\mathrm{MotSpc}}(M_X(Y)) = M_S(Y),$$

for smooth spectral schemes Y over X .

5.1.3. Let $p : X \rightarrow S$ be a smooth morphism. By Lemma 4.1.5, the functors p_{Spc}^* and p_{MotSpc}^* admit canonical symmetric monoidal structures, so that their respective left adjoints $p_{\sharp}^{\mathrm{Spc}}$ and $p_{\sharp}^{\mathrm{MotSpc}}$ admit colax symmetric monoidal structures.

If p is an *open immersion*, then these monoidal structures are strict:

Proposition 5.1.4. *Let $j : U \hookrightarrow X$ be a quasi-compact open immersion. Then the canonical colax symmetric monoidal structure on the functor $j_{\sharp}^{\mathrm{Spc}}$ (resp. $j_{\sharp}^{\mathrm{MotSpc}}$) is strict.*

Proof. It suffices to show that the canonical morphisms

$$j_{\sharp}^{\mathrm{Spc}}(\mathcal{F} \times_U \mathcal{G}) \rightarrow j_{\sharp}^{\mathrm{Spc}}(\mathcal{F}) \times_S j_{\sharp}^{\mathrm{Spc}}(\mathcal{G})$$

are invertible for all spaces \mathcal{F} and \mathcal{G} on U . Since $j_{\sharp}^{\mathrm{Spc}}$ commutes with colimits, and the cartesian product commutes with colimits in each argument, one reduces to the case of representables.

Then the claim follows from the fact that the fibred products $X \times_U Y$ and $X \times_S Y$ are canonically identified (since j is a monomorphism, i.e. its diagonal morphism is invertible). The claim for $j_{\sharp}^{\mathrm{MotSpc}}$ follows from the fact that motivic localization commutes with finite products. \square

5.1.5. Let $(f_{\alpha} : S_{\alpha} \rightarrow S)_{\alpha}$ be a Nisnevich covering family. Given a morphism of motivic spaces over S , the following proposition says that it is invertible if and only if its inverse image on each S_{α} is invertible:

Proposition 5.1.6 (Nisnevich separation). *Let S be a spectral scheme. For any Nisnevich covering family $(p_{\alpha} : S_{\alpha} \rightarrow S)_{\alpha}$, the family of inverse image functors $(p_{\alpha})_{\mathrm{MotSpc}}^* : \mathrm{MotSpc}^{\mathcal{E}^{\infty}}(S) \rightarrow \mathrm{MotSpc}^{\mathcal{E}^{\infty}}(S_{\alpha})$ is conservative.*

This is in fact true at the level of Nisnevich sheaves, which is what we will prove.

Proof. Let $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a morphism of Nisnevich sheaves on S , and suppose that the following condition holds:

(*) For each α , the morphism $(p_{\alpha})_{\mathrm{Nis}}^*(\mathcal{F}_1) \rightarrow (p_{\alpha})_{\mathrm{Nis}}^*(\mathcal{F}_2)$ is invertible.

The claim is that under this assumption, the morphism

$$\Gamma(X, \mathcal{F}_1) \rightarrow \Gamma(X, \mathcal{F}_2)$$

is invertible for every smooth spectral S -scheme X .

Since \mathcal{F}_i satisfy Nisnevich descent, it suffices to show that the morphism

$$(5.3) \quad \Gamma(X_\alpha, \mathcal{F}_1) \rightarrow \Gamma(X_\alpha, \mathcal{F}_2)$$

is invertible for each α , where X_α is the base change of X along p_α .

Since $h_S(X_\alpha) = (p_\alpha)_\#(p_\alpha)^*(h_S(X))$, we have by adjunction

$$\Gamma(X_\alpha, \mathcal{F}_i) = \Gamma(X, (p_\alpha)_\#(p_\alpha)^*\mathcal{F}_i)$$

for each α and i .

Hence the claim follows from the assumption $(*)$. \square

5.2. Smooth base change formulas.

5.2.1. Suppose we have a cartesian square

$$\begin{array}{ccc} T' & \xrightarrow{f'} & S' \\ \downarrow p' & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

of spectral schemes.

At the level of (motivic) spaces, pointed spaces, and spectra, there are canonical 2-morphisms

$$(5.4) \quad (p')_\#(f')^* \rightarrow f^*p_\#,$$

$$(5.5) \quad p^*f_* \rightarrow (f')_*(p')^*,$$

constructed in Paragraph B.4.

The following says that Spc and $\mathrm{MotSpc}^{\mathcal{E}^\infty}$ satisfy the *left base change property* along smooth morphisms (see *loc. cit.*):

Proposition 5.2.2. *If p and p' are smooth, then the 2-morphisms (5.4) and (5.5) are invertible at the level of spaces and motivic spaces.*

Proof. It suffices to consider (5.4); (5.5) is its right transpose.

For fibred spaces, we note that the functors in question commute with colimits, so that we may reduce to representable spaces, in which case the claim is obvious.

Similarly, for motivic spaces we may reduce to the case of motivic localizations of representable spaces. \square

5.2.3. Next we consider the case of pointed spaces. Then we have:

Proposition 5.2.4. *If p and p' are smooth, then the 2-morphisms (5.4) and (5.5) are invertible at the level of pointed spaces and pointed motivic spaces.*

Proof. By transposition it suffices to consider (5.4). Since the functors in question commute with colimits and with the functor $\mathcal{F} \mapsto \mathcal{F}_+$, the claim follows from Lemma A.5.4 (resp. Lemma A.5.12) and smooth base change for unpointed spaces (Proposition 5.2.2). \square

5.2.5. At the level of spectra, we have:

Proposition 5.2.6. *If p and p' are smooth, then the 2-morphisms (5.4) and (5.5) are invertible at the level of spectra and motivic spectra.*

Proof. This follows from Lemma A.6.11 (resp. Lemma A.6.21) and smooth base change for pointed spaces (Proposition 5.2.4). \square

5.3. Smooth projection formulas. Let $p : X \rightarrow S$ be a smooth morphism. Note that the symmetric monoidal functor p_{Spc}^* endows $\mathrm{Spc}(X)$ with a structure of $\mathrm{Spc}(S)$ -module arena.

The following verifies the *left projection formula* along smooth morphisms, in the sense of Paragraph B.4:

Proposition 5.3.1. *The functor $p_{\#}^{\mathrm{Spc}}$ (resp. $p_{\#}^{\mathrm{MotSpc}}$) lifts to a morphism of $\mathrm{Spc}(S)$ -module categories (resp. $\mathrm{MotSpc}^{\varepsilon\infty}(S)$ -module categories). In other words, there are canonical isomorphisms*

$$(5.6) \quad p_{\#}(\mathcal{G} \times_X p^*(\mathcal{F})) \rightarrow p_{\#}(\mathcal{G}) \times_S \mathcal{F}$$

and dually

$$(5.7) \quad \underline{\mathrm{Hom}}_S(p_{\#}(\mathcal{G}), \mathcal{F}) \rightarrow p_{\#}\underline{\mathrm{Hom}}_X(\mathcal{G}, p^*(\mathcal{F}))$$

for any fibred spaces (resp. motivic spaces) \mathcal{F} over S and \mathcal{G} over X .

We recall how to use the monoidal structure on p^* to construct the morphism (5.6):

The counit of the adjunction $(p_{\#}^{\mathrm{Spc}}, p_{\mathrm{Spc}}^*)$ induces a canonical morphism

$$\mathcal{G} \times_X p^*(\mathcal{F}) \rightarrow p^*p_{\#}(\mathcal{G}) \times_X p^*(\mathcal{F}) \xrightarrow{\sim} p^*(p_{\#}(\mathcal{G}) \times_S \mathcal{F})$$

which corresponds by adjunction to the morphism desired.

Proof. It suffices to show that the canonical morphism (5.6) is invertible. For fibred spaces, we may reduce to the case where the spaces \mathcal{F} and \mathcal{G} are representable, since the functions involved commute with colimits. In this case the claim is clear. The case of motivic spaces is similar, after reducing to the case of motivic localizations of representable spaces. \square

5.3.2. The following slightly more general formula, proved in exactly the same way, will also be useful:

Lemma 5.3.3. *Let $p : X \rightarrow S$ be a smooth morphism. Let \mathcal{G} be a fibred space (resp. motivic space) over X , and $\mathcal{F} \rightarrow \mathcal{F}'$ a morphism of fibred spaces (resp. motivic spaces) over S . Then there is a canonical isomorphism*

$$(5.8) \quad p_{\#}(\mathcal{G} \times_{p^*(\mathcal{F}')} p^*(\mathcal{F})) \xrightarrow{\sim} p_{\#}(\mathcal{G}) \times_{\mathcal{F}'} \mathcal{F}$$

of fibred spaces (resp. motivic spaces) over S .

5.3.4. Similarly we get smooth projection formulas for pointed spaces and spectra. As above, the following statements are equivalent to formulas of the form (5.6) and (5.7).

Proposition 5.3.5. *The functor $p_{\#}^{\mathrm{Spc}\bullet}$ (resp. $p_{\#}^{\mathrm{MotSpc}\bullet}$) lifts to a morphism of $\mathrm{Spc}(S)_{\bullet}$ -module categories (resp. $\mathrm{MotSpc}^{\varepsilon\infty}(S)_{\bullet}$ -module categories).*

Proof. This follows from Lemma A.5.4 (resp. Lemma A.5.12) and the smooth projection formula for unpointed spaces (Proposition 5.3.1). \square

Proposition 5.3.6. *The functor $p_{\#}^{\mathrm{Spt}}$ (resp. $p_{\#}^{\mathrm{MotSpt}}$) lifts to a morphism of $\mathrm{Spt}(S)$ -module categories (resp. $\mathrm{MotSpt}^{\varepsilon\infty}(S)$ -module categories).*

Proof. This follows from Lemma A.6.11 (resp. Lemma A.6.21) and the smooth projection formula for pointed spaces (Proposition 5.3.5). \square

6. FUNCTORIALITY ALONG CLOSED IMMERSIONS

In this section we study some properties of the functor i_* of direct image along a closed immersion i . In particular it turns out to admit a right adjoint $i^!$. We also state the localization theorem and deduce some of its interesting consequences, which include base change and projection formulas involving $i^!$.

6.1. The exceptional inverse image functor $i^!$.

6.1.1. Let $i : Z \hookrightarrow S$ be a closed immersion. Note that if the base change functor $\mathrm{Sm}_S \rightarrow \mathrm{Sm}_Z$ were topologically cocontinuous (see Paragraph C.1), then the direct image functor i_* on Nisnevich sheaves would commute with arbitrary small colimits. Though this is not quite true, we will show that this is true τ_{red} -locally (see Paragraph C.2), which will imply that i_* commutes with contractible colimits:

Proposition 6.1.2. *Let $i : Z \hookrightarrow S$ be a closed immersion. Then the direct image functor i_*^{MotSpc} commutes with contractible colimits.*

Proof. By Lemma C.2.6 it suffices to show that the base change functor $\mathrm{Sm}_S \rightarrow \mathrm{Sm}_Z$ is τ_{red} -locally cocontinuous. For this it suffices to check the condition $(\mathrm{COC}')_{\tau_{\mathrm{red}}}$ of Lemma C.2.4, which amounts to the following:

(*) For any smooth spectral S -scheme X and any Nisnevich covering sieve R' of X_Z , the sieve R of X generated by morphisms $X' \rightarrow X$ such that either (i) the empty sieve on X'_Z is Nisnevich-covering, or (ii) $X'_Z \rightarrow X_Z$ factors through R' , is Nisnevich-covering.

This condition follows directly from Proposition 2.13.2, which says that étale morphisms can be lifted (Zariski-locally) along i . \square

In particular:

Corollary 6.1.3. *The direct image functor $i_*^{\mathrm{MotSpc}\bullet}$ (resp. i_*^{MotSpt}) commutes with small colimits.*

By the adjoint functor theorem we have a right adjoint $i_{\mathrm{MotSpc}\bullet}^!$ (resp. $i_{\mathrm{MotSpt}}^!$), called the *exceptional inverse image* functor.

6.2. The localization theorem. In this paragraph, we will work in the category of motivic spaces, and will omit the decoration $\mathrm{MotSpc}^{\mathcal{E}^\infty}$ from the notation for simplicity.

6.2.1. Let $i : Z \hookrightarrow S$ be a closed immersion of spectral schemes with quasi-compact open complement $j : U \hookrightarrow S$. We deduce some immediate consequences of smooth base change in this situation.

Considering the commutative square

$$\begin{array}{ccc} U & \xlongequal{\quad} & U \\ \parallel & & \downarrow j \\ U & \xrightarrow{j} & S, \end{array}$$

which is cartesian because j is a monomorphism, we get:

Lemma 6.2.2. *For any quasi-compact open immersion $j : U \hookrightarrow S$, the canonical morphisms*

$$(6.1) \quad \text{id} \rightarrow j^* j_\#,$$

$$(6.2) \quad j^* j_* \rightarrow \text{id},$$

are invertible.

In other words, the functors $j_\#$ and j_* are fully faithful.

6.2.3. Considering the cartesian square

$$\begin{array}{ccc} \emptyset & \hookrightarrow & Z \\ \downarrow & & \downarrow i \\ U & \xhookrightarrow{j} & S, \end{array}$$

we get:

Lemma 6.2.4. *For any closed immersion $i : Z \hookrightarrow S$ with quasi-compact open complement $j : U \hookrightarrow S$, the canonical morphisms*

$$\emptyset_Z \rightarrow i^* j_\#(\mathcal{F}_U),$$

$$j^* i_*(\mathcal{F}_Z) \rightarrow \text{pt}_U,$$

are invertible, for \mathcal{F}_U (resp. \mathcal{F}_Z) a motivic space over U (resp. Z).

6.2.5. Consider the canonical commutative square

$$\begin{array}{ccc} j_\# j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\# j^* i_* i^*(\mathcal{F}) & \longrightarrow & i_* i^*(\mathcal{F}) \end{array}$$

for any motivic space \mathcal{F} over S .

By Lemma 6.2.4 this induces a canonical commutative square

$$(6.3) \quad \begin{array}{ccc} j_\# j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\#(\text{pt}_U) & \longrightarrow & i_* i^*(\mathcal{F}) \end{array}$$

which we call the *localization square* associated to the pair (i, j) .

The main theorem in this chapter is the following, due to [MV99] in the setting of classical algebraic geometry:

Theorem 6.2.6. *Let $i : Z \hookrightarrow S$ be a closed immersion with quasi-compact open complement $j : U \hookrightarrow S$. Then for every motivic space \mathcal{F} over S , the localization square (6.3) is cocartesian.*

The proof will occupy Sect. 7.

6.2.7. We can deduce from Theorem 6.2.6 a pointed version:

Corollary 6.2.8 (Localization). *Let $i : Z \hookrightarrow S$ be a closed immersion, with quasi-compact open complement $j : U \hookrightarrow S$. For any pointed motivic space (\mathcal{F}, s) over S , there is a canonical cofibre sequence*

$$(6.4) \quad j_\# j^*(\mathcal{F}, s) \rightarrow (\mathcal{F}, s) \rightarrow i_* i^*(\mathcal{F}, s).$$

and dually, a canonical fibre sequence

$$(6.5) \quad i_* i^! (\mathcal{F}, s) \rightarrow (\mathcal{F}, s) \rightarrow j_* j^* (\mathcal{F}, s)$$

of motivic spaces over S .

Proof. We want to show that the commutative square of pointed motivic spaces

$$\begin{array}{ccc} j_{\#} j^* (\mathcal{F}, x) & \longrightarrow & (\mathcal{F}, x) \\ \downarrow & & \downarrow \\ \mathrm{pt}_S^{\mathrm{Spc}\bullet} & \longrightarrow & i_* i^* (\mathcal{F}, x) \end{array}$$

is cocartesian.

Since the forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$ reflects contractible colimits (Lemma A.5.11), it suffices to show that the induced square of underlying motivic spaces

$$\begin{array}{ccc} j_{\#} j^* \mathcal{F} \sqcup_{j_{\#} j^* (\mathrm{pt}_S)} \mathrm{pt}_S & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathrm{pt}_S^{\mathrm{MotSpc}} & \longrightarrow & i_* i^* \mathcal{F}. \end{array}$$

is cocartesian.

Consider the composite square

$$\begin{array}{ccccc} j_{\#} j^* \mathcal{F} & \longrightarrow & (j_{\#} j^* \mathcal{F}) \sqcup_{j_{\#} j^* (\mathrm{pt}_S)} \mathrm{pt}_S & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ j_{\#} j^* (\mathrm{pt}_S) & \longrightarrow & \mathrm{pt}_S & \longrightarrow & i_* i^* \mathcal{F}. \end{array}$$

which is cocartesian by Theorem 6.2.6.

Since the left-hand square is evidently cocartesian, it follows that the right-hand square is also cocartesian. \square

6.2.9. Similarly we also deduce localization for motivic spectra:

Corollary 6.2.10. *Let $i : Z \hookrightarrow S$ be a closed immersion, with quasi-compact open complement $j : U \hookrightarrow S$. For any motivic spectrum \mathbb{E} over S , there is a canonical cofibre sequence*

$$(6.6) \quad j_{\#} j^* (\mathbb{E}) \rightarrow \mathbb{E} \rightarrow i_* i^* (\mathbb{E}),$$

and dually a fibre sequence

$$(6.7) \quad i_* i^! (\mathbb{E}) \rightarrow \mathbb{E} \rightarrow j_* j^* (\mathbb{E}),$$

of motivic spectra over S .

Proof. It suffices to show the first sequence is a cofibre sequence. Since the functors in question commute with small colimits, Proposition 3.6.3 allows us to reduce to the case of pointed motivic spaces, which is Corollary 6.2.8. \square

6.2.11. An immediate corollary of Theorem 6.2.6 is:

Corollary 6.2.12. *Let $i : Z \hookrightarrow S$ be a closed immersion with quasi-compact open complement. Then the direct image functor i_*^{MotSpc} (resp. $i_*^{\text{MotSpc}\bullet}$, i_*^{MotSpt}) is fully faithful.*

Proof. The claims for $i_*^{\text{MotSpc}\bullet}$ and i_*^{MotSpt} follow directly from that of i_*^{MotSpc} .

Considering the localization square for $i_*(\mathcal{F})$, we see that the canonical morphism $i_*i^*i_* \rightarrow i_*$ is invertible. Hence it suffices to show that i_* is conservative.

For this, let $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a morphism of motivic spaces over Z such that $i_*(\varphi)$ is invertible. To show that φ is invertible, it suffices to show that

$$\Gamma(X, \mathcal{F}_1) \rightarrow \Gamma(X, \mathcal{F}_2)$$

is invertible for each smooth spectral Z -scheme X .

By Proposition 2.13.2, we may assume that X is the base change of a smooth spectral S -scheme Y . In this case the claim follows by assumption, since $\Gamma(X, \mathcal{F}_i) = \Gamma(Y, i_*(\mathcal{F}_i))$ for each i , by adjunction. \square

6.3. Closed base change formula.

6.3.1. Let Θ be a cartesian square

$$(6.8) \quad \begin{array}{ccc} X_Z & \xrightarrow{k} & X \\ \downarrow g & & \downarrow f \\ Z & \xrightarrow{i} & S, \end{array}$$

of spectral schemes, with i and k closed immersions with quasi-compact open complements.

At the level of motivic spaces, there is a canonical 2-morphism

$$(6.9) \quad k_*g^* \rightarrow f^*i_*$$

constructed in Paragraph B.5.

The following says that $\text{MotSpc}^{\mathcal{E}^\infty}$ satisfies the *right base change property* along closed immersions (see *loc. cit.*):

Corollary 6.3.2. *The 2-morphism (6.9) is invertible at the level of motivic spaces.*

Proof. This follows by considering the localization squares associated to the closed immersions j and k , respectively, and using the smooth base change formula (Proposition 5.2.2). \square

6.3.3. In the pointed setting, the functor i_* admits a right adjoint $i^!$ (Corollary 6.1.3), so we obtain another 2-morphism by right transposition from (6.9). Hence we have:

Corollary 6.3.4. *Given a cartesian square of the form (6.8), the canonical 2-morphisms*

$$(6.10) \quad k_*g^* \rightarrow f^*i_*$$

$$(6.11) \quad i^!f_* \rightarrow g_*k^!$$

are invertible at the level of pointed motivic spaces.

6.3.5. At the level of spectra, we have:

Corollary 6.3.6. *Given a cartesian square of the form (6.8), the canonical 2-morphisms*

$$(6.12) \quad k_* g^* \rightarrow f^* i_*$$

$$(6.13) \quad i^! f_* \rightarrow g_* k^!$$

are invertible at the level of motivic spectra.

6.4. Closed projection formula. Let $i : Z \hookrightarrow S$ be a closed immersion with quasi-compact open complement. Note that the symmetric monoidal functor i_{Spc}^* endows $\text{MotSpc}^{\mathcal{E}\infty}(Z)$ with a structure of $\text{MotSpc}^{\mathcal{E}\infty}(S)$ -module category.

The following verifies the *right projection formula* along closed immersions, in the sense of Paragraph B.5:

Proposition 6.4.1. *The functor i_*^{MotSpc} lifts to a morphism of $\text{MotSpc}^{\mathcal{E}\infty}(S)$ -module categories. In other words, there are canonical isomorphisms*

$$(6.14) \quad i_*(\mathcal{G} \times_Z i^*(\mathcal{F})) \rightarrow i_*(\mathcal{G}) \times_S \mathcal{F}$$

for any motivic spaces \mathcal{F} over S and \mathcal{G} over Z , and dually

$$(6.15) \quad i^! \underline{\text{Hom}}_S(\mathcal{G}, \mathcal{F}) \rightarrow \underline{\text{Hom}}_Z(i^* \mathcal{G}, i^! \mathcal{F})$$

for any motivic spaces \mathcal{F} and \mathcal{G} over S .

Proof. The second isomorphism is the right transpose of the first. The first follows from the localization theorem (Corollary 6.2.8) and the smooth projection formula. \square

6.4.2. Similarly we get closed projection formulas for pointed motivic spaces and spectra. As above, the following statements are equivalent to formulas of the form (6.14) and (6.15). The proofs are completely analogous to those of Proposition 5.3.5 and Proposition 5.3.6.

Corollary 6.4.3. *The functor $i_*^{\text{MotSpc}\bullet}$ lifts to a morphism of $\text{MotSpc}^{\mathcal{E}\infty}(S)_\bullet$ -module categories.*

At the level of spectra, we have:

Corollary 6.4.4. *The functor i_*^{MotSpt} lifts to a morphism of $\text{MotSpt}^{\mathcal{E}\infty}(S)$ -module categories.*

6.5. Smooth-closed base change formula.

6.5.1. Let Θ be a cartesian square of spectral schemes

$$(6.16) \quad \begin{array}{ccc} X_Z & \xrightarrow{k} & X \\ \downarrow q & & \downarrow p \\ Z & \xrightarrow{i} & S, \end{array}$$

where i and j are closed immersions with quasi-compact open complements, and p and q are smooth.

There are canonical 2-morphisms

$$(6.17) \quad p_{\#} k_* \rightarrow i_* q_{\#}$$

$$(6.18) \quad q^* i^! \rightarrow k^! p^*$$

at the level of pointed motivic spaces, constructed in Paragraph B.6.

The following verifies the *bidirectional base change property* with respect to smooth morphisms and closed immersions:

Corollary 6.5.2 (Smooth-closed base change). *Given a cartesian square of the form (6.16), the 2-morphisms (6.17) and (6.18) are invertible at the level of pointed motivic spaces.*

Proof. The second transformation is obtained by passing to right adjoints from the first. For the first, it suffices by Corollary 6.2.12 it suffices to show that the transformation

$$p_{\sharp} k_{*} k^{*} \rightarrow i_{*} q_{\sharp} k^{*},$$

obtained by pre-composition with k^{*} , is invertible. This follows directly from Corollary 6.2.8 and smooth base change. \square

7. THE LOCALIZATION THEOREM

This section is dedicated to the proof of the localization theorem (see Paragraph 6.2).

Throughout the section, we let $i : Z \hookrightarrow S$ be a closed immersion of spectral schemes, such that the complementary open immersion $j : U \hookrightarrow S$ is quasi-compact.

7.1. The space $h_S(X, t)$.

7.1.1. Given a smooth spectral S -scheme X , let $X_U := X \times_S U$ denote its base change along j , and $X_Z := X \times_S Z$ its base change along i .

We will write $h_S^Z(X)$ for the space over S defined by the cocartesian square

$$(7.1) \quad \begin{array}{ccc} h_S(X_U) & \longrightarrow & h_S(X) \\ \downarrow & & \downarrow \\ h_S(U) & \longrightarrow & h_S^Z(X). \end{array}$$

Note that there is a canonical isomorphism

$$(7.2) \quad i_{\mathrm{Spc}}^{*}(h_S^Z(X)) = h_Z(X_Z)$$

of spaces over Z , since i_{Spc}^{*} commutes with colimits.

Since colimits in $\mathrm{Spc}(S)$ are computed section-wise, we can describe the spaces of sections of $h_S^Z(X)$ explicitly:

Lemma 7.1.2. *Let Y be a smooth spectral S -scheme. If Y_Z is the empty spectral scheme, then the space $\Gamma(Y, h_S^Z(X))$ is contractible. Otherwise, there is a canonical isomorphism of spaces*

$$(7.3) \quad \Gamma(Y, h_S^Z(X)) = \Gamma(Y, h_S(X)) = \mathrm{Maps}_S(Y, X).$$

7.1.3. Let $p : X \rightarrow S$ be a smooth morphism. Let $t : Z \hookrightarrow X$ be an S -morphism, i.e. a partially defined section of p .

Consider the canonical morphism

$$(7.4) \quad \varepsilon : h_S^Z(X) \rightarrow i_{*}^{\mathrm{Spc}} i_{\mathrm{Spc}}^{*}(h_S^Z(X)) = i_{*}^{\mathrm{Spc}}(h_Z(X_Z))$$

induced by the counit of the adjunction $(i_{\mathrm{Spc}}^{*}, i_{*}^{\mathrm{Spc}})$.

The morphism t corresponds by adjunction to a morphism $\tau : h_S(S) \rightarrow i_{*}^{\mathrm{Spc}}(h_S(X_Z))$. We define a space $h_S(X, t)$ over S as the fibre of ε at the point τ , so that we have a cartesian square

$$\begin{array}{ccc} h_S(X, t) & \longrightarrow & h_S^Z(X) \\ \downarrow & & \downarrow \varepsilon \\ h_S(S) & \xrightarrow{\tau} & i_{*}^{\mathrm{Spc}}(h_S(X_Z)) \end{array}$$

of spaces over S .

7.1.4. Over spectral S -schemes that do not vanish on Z , sections of $h_S(X, t)$ are S -sections of X extending t . More precisely (recall that limits in $\mathrm{Spc}(S)$ are computed section-wise):

Lemma 7.1.5. *Let Y be a smooth spectral S -scheme. If Y_Z is the empty spectral scheme, then the space $\Gamma(Y, h_S(X, t))$ is contractible. Otherwise, $\Gamma(Y, h_S(X, t))$ is canonically identified with the fibre of the restriction map*

$$\mathrm{Maps}_S(Y, X) \rightarrow \mathrm{Maps}_Z(Y_Z, X_Z)$$

at the point defined by the composite $Y_Z \rightarrow Z \xrightarrow{t} X_Z$.

In other words, points of the space $\Gamma(Y, h_S(X, t))$ are pairs (f, α) , with $f : Y \rightarrow X$ an S -morphism and α a commutative triangle

$$\begin{array}{ccc} Y_Z & \xrightarrow{f_Z} & X_Z \\ \downarrow & \nearrow t & \\ Z & & \end{array}$$

7.1.6. If p is a smooth morphism, then since p_{Spc}^* commutes with both limits and colimits, we have:

Lemma 7.1.7. *Let X be a smooth spectral S -scheme and $t : Z \hookrightarrow X$ an S -morphism. If $p : T \rightarrow S$ is a smooth morphism, then there is a canonical isomorphism of fibred spaces*

$$p_{\mathrm{Spc}}^*(h_S(X, t)) = h_T(X_T, t_T),$$

where $t_T : Z_T \hookrightarrow X_T$ is obtained from t by base change along p .

7.1.8. Our main result about the fibred space $h_S(X, t)$ is as follows:

Proposition 7.1.9. *Let X be an affine smooth spectral S -scheme. Then for every S -morphism $t : Z \hookrightarrow X$, the space $h_S(X, t)$ is motivically contractible, i.e. the morphism $h_S(X, t) \rightarrow \mathrm{pt}_S$ is a motivic equivalence.*

The proof will occupy the rest of this section.

7.1.10. We first consider the case of vector bundles:

Lemma 7.1.11. *Let E be a vector bundle over S with zero section $s : S \hookrightarrow E$. Then the space $h_S(E, s_Z)$ is motivically contractible, where $s_Z : Z \hookrightarrow E_Z$ denotes the base change of s along $i : Z \hookrightarrow S$.*

Proof. It suffices to construct an \mathbf{I} -homotopy inverse to the unique morphism

$$\varphi : h_S(E, s_Z) \rightarrow h_S(S).$$

The zero section induces a canonical morphism

$$h_S(S) \xrightarrow{s} h_S(E) \rightarrow h_S^Z(E),$$

which induces a canonical morphism

$$\psi : h_S(S) \rightarrow h_S(E, s_Z).$$

It remains to define an \mathbf{I} -homotopy

$$\vartheta : h_S(S \times \mathbf{I}) \times_S h_S(E, s_Z) \rightarrow h_S(E, s_Z)$$

between the identity and the composite $\psi \circ \varphi$. For each smooth spectral S-scheme Y with $Y_Z \neq \emptyset$, define

$$\Gamma(Y, \vartheta) : \Gamma(Y, h_S(S \times \mathbf{I})) \times \Gamma(Y, h_S(E, s_Z)) \rightarrow \Gamma(Y, h_S(E, s_Z))$$

by the assignment

$$(a : Y \rightarrow S \times \mathbf{I}, f : Y \rightarrow E) \mapsto (a \cdot f : Y \rightarrow E).$$

It is clear that this defines the \mathbf{I} -homotopy desired. \square

7.2. Étale base change.

7.2.1. The assignment $(X, t) \mapsto h_S(X, t)$ is functorial in the following sense.

Let (X, t) and (X', t') be pairs, with X (resp. X') a smooth spectral S-scheme, and $t : Z \hookrightarrow X$ (resp. $t' : Z \hookrightarrow X'$) a partially defined section. Suppose $f : X' \rightarrow X$ is an S-morphism such that the square

$$\begin{array}{ccc} Z & \xrightarrow{t'} & X'_Z \\ \parallel & & \downarrow \\ Z & \xrightarrow{t} & X_Z \end{array}$$

is cartesian. Then there is a canonical morphism of spaces over S

$$(7.5) \quad h_S(X', t') \rightarrow h_S(X, t).$$

Lemma 7.2.2. *Suppose that (X, t) and (X', t') are pairs as above. Let $p : X' \rightarrow X$ be an étale morphism, such that the above square is cartesian. Then the induced morphism*

$$\varphi : h_S(X', t') \rightarrow h_S(X, t)$$

is a Nisnevich-local equivalence.

The claim is that the induced morphism of Nisnevich sheaves $L_{\text{Nis}}(\varphi)$ is invertible. It suffices to show that it is 0-truncated (i.e. its diagonal is a monomorphism) and 0-connected (i.e. it is an effective epimorphism and so is its diagonal).

7.2.3. *Proof of Lemma 7.2.2, step 1.* To show that $L_{\text{Nis}}(\varphi)$ is 0-truncated, it suffices to show that φ is 0-truncated (since L_{Nis} is exact). For this, it suffices to show that for each smooth spectral S-scheme Y , the induced morphism of spaces of Y-sections

$$\Gamma(Y, \varphi) : \Gamma(Y, h_S^Z(X', t')) \rightarrow \Gamma(Y, h_S^Z(X, t))$$

is 0-truncated.

We may assume Y_Z is not empty; then this is the morphism induced on fibres in the diagram

$$\begin{array}{ccccc} \Gamma(Y, h_S^Z(X', t')) & \longrightarrow & \text{Maps}_S(Y, X') & \longrightarrow & \text{Maps}_Z(Y_Z, X'_Z) \\ \downarrow \text{dashed} & & \downarrow & & \downarrow \\ \Gamma(Y, h_S^Z(X, t)) & \longrightarrow & \text{Maps}_S(Y, X) & \longrightarrow & \text{Maps}_Z(Y_Z, X_Z) \end{array}$$

Note that the two right-hand vertical morphisms are 0-truncated: p is itself 0-truncated since it is étale, and since the Yoneda embedding commutes with limits, the induced morphism $h_S(X') \rightarrow h_S(X)$ is also 0-truncated. It follows that the left-hand vertical morphism is also 0-truncated for each Y , and therefore so is φ .

7.2.4. *Proof of Lemma 7.2.2, step 2.* To show that $L_{\text{Nis}}(\varphi)$ is an effective epimorphism, it suffices to show that for each smooth spectral S-scheme Y (with Y_Z not empty), any Y -section of $h_S^Z(X, t)$ can be lifted Nisnevich-locally along φ .

Let f be a Y -section of $h_S^Z(X, t)$, i.e. a morphism $f : Y \rightarrow X$ together with an isomorphism between f_Z and the composite $Y_Z \rightarrow Z \xrightarrow{t} X_Z$. Let $q : Y' \rightarrow Y$ denote the base change of $p : X' \rightarrow X$ along f :

$$\begin{array}{ccc} Y' & \xrightarrow{q} & Y \\ \downarrow g & & \downarrow f \\ X' & \xrightarrow{p} & X. \end{array}$$

Then note that

$$\begin{array}{ccc} q^{-1}(Y_U) & \hookrightarrow & Y' \\ \downarrow & & \downarrow q \\ Y_U & \hookrightarrow & Y \end{array}$$

is a Nisnevich square. Indeed, the closed immersion $Y_Z \hookrightarrow Y$ is complementary to $Y_U \hookrightarrow Y$, and it is clear that $q^{-1}(Y_Z) \rightarrow Y_Z$ is invertible because in the diagram

$$\begin{array}{ccc} q^{-1}(Y_Z) & \longrightarrow & Y_Z \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\text{id}_Z} & Z \\ \downarrow t & & \downarrow t' \\ X'_Z & \xrightarrow{pz} & X_Z \end{array}$$

the lower square and the composite square are cartesian, and hence so is the upper square.

Hence it suffices to show that the restriction of f to either component of this Nisnevich cover lifts to $h_S^Z(X', t')$. The restriction $f|_{Y'}$ lifts to a section of $h_S^Z(X', t')$ given by $g : Y' \rightarrow X'$. The restriction $f|_{Y_U}$ admits a lift trivially: since $(Y_U) \times_S Z = \emptyset$, the spaces $h_S^Z(X, t)(Y_U)$ and $h_S^Z(X', t')(Y_U)$ are both contractible.

7.2.5. *Proof of Lemma 7.2.2, step 3.* It remains to show that the diagonal $\Delta_{L_{\text{Nis}}(\varphi)}$ of $L_{\text{Nis}}(\varphi)$ is an effective epimorphism, or equivalently that $L_{\text{Nis}}(\Delta_\varphi)$ is.

For each smooth spectral S-scheme Y , the diagonal induces a morphism of spaces

$$\Gamma(Y, h_S^Z(X', t')) \rightarrow \Gamma(Y, h_S^Z(X', t')) \times_{\Gamma(Y, h_S^Z(X, t))} \Gamma(Y, h_S^Z(X', t')).$$

It suffices to show that for each Y (with Y_Z not empty), any Y -section of the target lifts Nisnevich-locally to a Y -section of the source. Choose a section of the target, given by two Y -sections $f : Y \rightarrow X'$ and $g : Y \rightarrow X'$, and an identification $\alpha : p \circ f \rightarrow p \circ g$.

Let $Y_0 \hookrightarrow Y$ denote the open immersion defined as the equalizer of the pair (f, g) . Note that the closed immersion $Y_Z \hookrightarrow Y$ factors through Y_0 . Hence the open immersions $Y_0 \hookrightarrow Y$ and $Y_U \hookrightarrow Y$ form a Zariski cover of Y . It is clear that the Y -section (f, g, α) lifts after restriction to Y_0 by definition, and after restriction to Y_U since $Y_U \times_S Z = \emptyset$, so the claim follows.

7.3. Reduction to the case of vector bundles. We reduce to the case of vector bundles in two steps: first, we show that partial sections of smooth spectral S-schemes can be lifted Nisnevich-locally to globally defined sections; second, we show using étale base change that

smooth spectral S -schemes with globally defined sections can be replaced by their conormal bundles.

7.3.1. The following lemma will allow us to reduce to the situation where the Z -section t lifts to an S -section $s : S \hookrightarrow X$.

Lemma 7.3.2. *Let $p : X \rightarrow S$ be a smooth morphism. Given an S -morphism $t : Z \hookrightarrow X$, there exists a Nisnevich square*

$$(7.6) \quad \begin{array}{ccc} Y_U & \hookrightarrow & Y \\ \downarrow & & \downarrow q \\ U & \xhookrightarrow{j} & S \end{array}$$

such that q factors through p .

Proof. We will construct a commutative square

$$(7.7) \quad \begin{array}{ccc} Z & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ X_Z & \hookrightarrow & X. \end{array}$$

with the following properties:

- (i) The induced square of underlying classical schemes

$$\begin{array}{ccc} Z_{\text{cl}} & \hookrightarrow & Y_{\text{cl}} \\ \downarrow & & \downarrow \\ (X_Z)_{\text{cl}} & \hookrightarrow & X_{\text{cl}} \end{array}$$

is cartesian.

- (ii) The composite morphism $Y \rightarrow X \rightarrow S$ is étale.

Given such a square (7.7), it is clear that we get a Nisnevich square (7.6) as claimed, by taking q to be the composite $Y \rightarrow X \rightarrow S$: indeed, the closed immersion $Z_{\text{cl}} \hookrightarrow S$ is complementary to j , and the squares

$$\begin{array}{ccccc} Z_{\text{cl}} & \hookrightarrow & Y_{\text{cl}} & \hookrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow q \\ Z_{\text{cl}} & \hookrightarrow & S_{\text{cl}} & \hookrightarrow & S \end{array}$$

are cartesian: the left-hand one by (i), and the right-hand one by flatness of q , which is implied by (ii).

In the classical case, the existence of the square (7.7) is known (this is a non-equivariant version of [Hoy15b, Thm. 2.21], for instance).

Hence one obtains a cartesian square

$$\begin{array}{ccc} Z_{\text{cl}} & \dashrightarrow & Y_0 \\ \downarrow & & \downarrow \\ (X_Z)_{\text{cl}} & \hookrightarrow & X_{\text{cl}} \end{array}$$

of classical schemes. Then one defines Y by the cocartesian square of closed immersions

$$\begin{array}{ccc} Z_{\text{cl}} & \hookrightarrow & Y_0 \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & Y. \end{array}$$

By Lemma 2.12.2 this exists, and the morphism $Y_0 \hookrightarrow Y$ is a closed immersion identifying Y_0 with the classical scheme underlying Y . The existence of the desired commutative square

$$\begin{array}{ccc} Z & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X_Z & \hookrightarrow & X \end{array}$$

follows by construction. \square

7.3.3. Next, we show that an S -section $s : S \hookrightarrow X$ may be approximated *globally* by the zero section of its conormal bundle.

Any closed immersion of smooth S -schemes is *quasi-smooth* in the sense of [AG15]. Such morphisms admit local presentations as base changes of zero sections of vector bundles (see Prop. 2.1.10 of *loc. cit.*). In our case, our closed immersion s is not only quasi-smooth, but admits a smooth retract $p : X \rightarrow S$. This allows us to construct the global presentation we want.

More precisely:

Lemma 7.3.4. *Let $p : X \rightarrow S$ be a smooth affine morphism. If p admits a section $s : S \hookrightarrow X$, then there exists an S -morphism $q : X \rightarrow \mathbf{N}_s^*$ to the conormal bundle of s satisfying the following conditions:*

(i) *The commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{s} & X \\ \parallel & & \downarrow q \\ S & \xrightarrow{z} & \mathbf{N}_s^* \end{array}$$

is cartesian, where t denotes the zero section.

(ii) *The morphism s factors through an open immersion $j_0 : X_0 \hookrightarrow X$ with $q \circ j_0$ étale.*

Proof. The conormal bundle \mathbf{N}_s^* is by definition the vector bundle associated to the shifted cotangent sheaf $\mathcal{N}_s^* = \mathcal{T}^*(S/X)[-1]$, which is a locally free \mathcal{O}_S -module of finite rank.

Consider the closed immersion $s_{\text{cl}} : S_{\text{cl}} \hookrightarrow X_{\text{cl}}$ of underlying classical schemes. Let \mathcal{I} denote its defining quasi-coherent sheaf of ideals, and $\mathcal{N}_{s_{\text{cl}}} = i^*(\mathcal{N}_s)$ its conormal bundle, where i is the closed immersion $S_{\text{cl}} \hookrightarrow S$.

The epimorphism $(p_{\text{cl}})_*(\mathcal{I}) \rightarrow \mathcal{N}_{s_{\text{cl}}}^*$ admits a section, since $\mathcal{N}_{s_{\text{cl}}}^*$ is projective, so that one obtains a morphism $\mathcal{N}_{s_{\text{cl}}}^* \rightarrow (p_{\text{cl}})_*(\mathcal{O}_{X_{\text{cl}}})$. This lifts to a morphism $\mathcal{N}_s^* \rightarrow p_*(\mathcal{O}_X)$, corresponding to a morphism of \mathcal{O}_S -algebras

$$\varphi : \text{Free}_{\mathcal{O}_S}(\mathcal{N}_s^*) \rightarrow p_*(\mathcal{O}_X).$$

Then it is clear that the commutative square of \mathcal{O}_S -algebras

$$\begin{array}{ccc} \mathrm{Free}_{\mathcal{O}_S}(\mathcal{N}_s^*) & \xrightarrow{\zeta} & \mathcal{O}_S \\ \downarrow \varphi & & \parallel \\ p_*(\mathcal{O}_X) & \xrightarrow{\sigma} & \mathcal{O}_S \end{array}$$

is cocartesian. Here ζ corresponds to the zero section z and σ to s .

We let $q : X \rightarrow \mathbf{N}_s^*$ be the morphism of spectral S -schemes corresponding to φ .

For (ii), let j_0 be the étale locus of q . To show that s factors through j_0 , it is sufficient to note that $s^*(\mathcal{T}^*(X/\mathbf{N}_s^*)) = \mathcal{T}_{S/S}^* = 0$ by (i). \square

7.4. Motivic contractibility of $\mathrm{h}_S(X, t)$. In this paragraph we prove Proposition 7.1.9.

7.4.1. Let X be an affine smooth spectral S -scheme, with structural morphism $p : X \rightarrow S$. Recall the statement of Proposition 7.1.9: we want to show that for any S -morphism $t : Z \hookrightarrow X$, the fibred space $\mathrm{h}_S(X, t)$ is motivically contractible.

7.4.2. By Lemma 7.3.2 there exists a Nisnevich square

$$(7.8) \quad \begin{array}{ccc} Y_U & \hookrightarrow & Y \\ \downarrow & & \downarrow q \\ U & \xrightarrow{j} & S \end{array}$$

where q factors through $p : X \rightarrow S$. It suffices then by the Nisnevich separation property (Proposition 5.1.6) to show that $j^* \mathrm{h}_S(X, t)$ and $q^* \mathrm{h}_S(X, t) = \mathrm{h}_Y(Y \times_S X, t')$ are contractible, where $t' : Y_Z \hookrightarrow (Y \times_S X)_Z$ is the base change of t .

7.4.3. The case of $j^* \mathrm{h}_S(X, t)$ is clear, since j is complementary to $i : Z \hookrightarrow S$.

7.4.4. For $q^* \mathrm{h}_S(X, t)$, note that by construction there exists a section $t'' : Y \hookrightarrow Y \times_S X$ which lifts t' (since q factors through X):

$$\begin{array}{ccc} (Y \times_S X)_Z & \hookrightarrow & Y \times_S X \\ \uparrow t' & & \uparrow t'' \\ Y_Z & \hookrightarrow & Y \end{array}$$

Hence by Lemma 7.3.4, Lemma 7.2.2 and Lemma 7.1.11, we have motivic equivalences

$$\mathrm{h}_S(Y \times_S X, t') = \mathrm{h}_S(\mathbf{N}_{t''}^*, z) = \mathrm{h}_S(S),$$

where $\mathbf{N}_{t''}^*$ is the conormal bundle, and z is the base change of its zero section.

7.5. Proof of the localization theorem. We conclude this section by proving the localization theorem, using Proposition 7.1.9.

Recall that our goal is to show that the canonical morphism

$$(7.9) \quad \mathcal{F} \bigsqcup_{j_\# j^*(\mathcal{F})} \mathrm{M}_S(U) \rightarrow i_* i^*(\mathcal{F})$$

is invertible for each motivic space \mathcal{F} over S .

7.5.1. First, note that we may reduce to the case where \mathcal{F} is a motivic localization $M_S(X)$ of an affine smooth spectral S -scheme X . Indeed, we have seen that the category $\text{MotSpc}^{\varepsilon_\infty}(S)$ is generated under sifted colimits by such objects (Proposition 3.3.5) and that each of the functors j_\sharp , j^* , i^* , and i_* commutes with contractible colimits (Proposition 6.1.2).

In this case the morphism (7.9) is canonically identified with the morphism

$$(7.10) \quad M_S(X) \bigsqcup_{M_S(X_U)} M_S(U) \rightarrow i_* M_S(X_Z)$$

where we write $X_U = X \times_S U$ and $X_Z = X \times_S Z$.

7.5.2. Note that the source of the morphism (7.10) is the motivic localization of the space $h_S^Z(X)$, and that the target $i_*^{\text{MotSpc}}(M_Z(X_Z))$ is the motivic localization of $i_*^{\text{Spc}}(h_Z(X_Z))$.

Hence it suffices to show that the morphism

$$(7.11) \quad h_S^Z(X) \rightarrow i_*^{\text{Spc}} h_Z(X_Z)$$

is a motivic equivalence.

7.5.3. By universality of colimits, it suffices to show that for every smooth spectral S -scheme Y and every morphism $h_S(Y) \rightarrow i_*^{\text{Spc}} h_Z(X_Z)$, corresponding to an S -morphism $t : Z \rightarrow X$, the base change

$$(7.12) \quad h_S^Z(X) \times_{i_*^{\text{Spc}} h_Z(X_Z)} h_S(Y) \rightarrow h_S(Y)$$

is invertible.

7.5.4. Let $p : Y \rightarrow S$ be the structural morphism of Y . Then since $h_S(Y) = p_\sharp^{\text{Spc}} h_Y(Y)$, one sees that (7.12) is identified, by the smooth projection formula (Lemma 5.3.3), with a morphism

$$(7.13) \quad p_\sharp^{\text{Spc}}(p_{\text{Spc}}^* h_S^Z(X) \times_{p_{\text{Spc}}^* i_*^{\text{Spc}} h_Z(X_Z)} h_Y(Y)) \rightarrow p_\sharp^{\text{Spc}} h_Y(Y).$$

7.5.5. Note that we have $p^* i_* = k_* q^*$ (Proposition 5.2.2), where k (resp. q) is the base change of i (resp. p) along p (resp. i). Hence the morphism (7.13) is identified with the image by p_\sharp of

$$(7.14) \quad h_Y^{Y_Z}(X \times_S Y) \times_{k_*^{\text{Spc}} h_{Y_Z}((X \times_S Y)_Z)} h_Y(Y) \rightarrow h_Y(Y).$$

7.5.6. The source of the morphism (7.14) is nothing else than the space $h_Y(X \times_S Y, t_Y)$, where $t_Y : Z \times_S Y \rightarrow X \times_S Y$ is the base change of t along p . Hence we conclude by Proposition 7.1.9.

APPENDIX A. MOREL–VOEVODSKY HOMOTOPY THEORY

In this section we collect some generalities about Morel–Voevodsky homotopy theory in an abstract setting. Given any essentially small $(\infty, 1)$ -category \mathbf{C} , equipped with some additional structure, we construct an unstable homotopy category associated to \mathbf{C} . We also axiomatize the process of stabilization. We try to do all this as generally as possible, in order to encompass all possible versions of Morel–Voevodsky homotopy theory, including equivariant motivic homotopy theory as developed in [Hoy15b], and even noncommutative motivic homotopy theory as developed in [Rob15].

Let \mathbf{C} be an essentially small $(\infty, 1)$ -category. Throughout this section we will write $\text{PSh}(\mathbf{C})$ for the $(\infty, 1)$ -category of presheaves of spaces on \mathbf{C} , i.e. functors $(\mathbf{C})^{\text{op}} \rightarrow \text{Spc}$. We will write $h_{\mathbf{C}} : \mathbf{C} \hookrightarrow \text{PSh}(\mathbf{C})$ for the fully faithful functor defined by the Yoneda embedding.

A.1. Excision structures. The notion of excision structure is a slight variation on *cd-structures* studied by Voevodsky [Voe10].

A.1.1. A *pre-excision structure* on \mathbf{C} is an essentially small set \mathbb{E} of cartesian squares in \mathbf{C} .

We may consider the following axioms on \mathbb{E} :

(EXC0) There exists an initial object $\emptyset_{\mathbf{C}}$ of \mathbf{C} such that every morphism $c \rightarrow \emptyset_{\mathbf{C}}$ is invertible.

(EXC1) The set \mathbb{E} is stable under base change. That is, for any square Q in \mathbb{E} , the base change along any morphism in \mathbf{C} exists and belongs to \mathbb{E} .

Definition A.1.2. We say that \mathbb{E} is an excision structure if it satisfies the axioms (EXC0) and (EXC1).

A.1.3. Let \mathbb{E} be a pre-excision structure on \mathbf{C} . We define:

Definition A.1.4. A presheaf \mathcal{F} on \mathbf{C} is \mathbb{E} -excisive if it satisfies the following conditions:

- (1) The space $\mathcal{F}(\emptyset_{\mathbf{C}})$ is contractible.
- (2) For any cartesian square $Q \in \mathbb{E}$ of the form

$$(A.1) \quad \begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow g \\ c' & \xrightarrow{f} & c, \end{array}$$

the induced commutative square of spaces

$$\begin{array}{ccc} \mathcal{F}(c) & \longrightarrow & \mathcal{F}(c') \\ \downarrow & & \downarrow \\ \mathcal{F}(d) & \longrightarrow & \mathcal{F}(d') \end{array}$$

is cartesian.

A.1.5. Note that we have:

Lemma A.1.6. The condition of \mathbb{E} -excision is stable by filtered colimits.

Proof. Let \mathcal{F} be the colimit of a filtered diagram $(\mathcal{F}_\alpha)_\alpha$, where each \mathcal{F}_α is \mathbb{E} -excisive. The condition (1) follows from the fact that filtered colimits of contractible spaces are contractible. The condition (2) follows from the fact that finite limits commute with filtered colimits in $\mathbf{PSh}(\mathbf{C})$. \square

A.1.7. Let $\mathbf{PSh}_{\mathbb{E}}(\mathbf{C})$ denote the full sub- $(\infty, 1)$ -category of $\mathbf{PSh}(\mathbf{C})$ spanned by \mathbb{E} -excisive presheaves.

Note that this is a left localization of $\mathbf{PSh}(\mathbf{C})$ at the (essentially small) set containing the canonical morphism

$$e : \emptyset_{\mathbf{PSh}(\mathbf{C})} \rightarrow h_{\mathbf{C}}(\emptyset_{\mathbf{C}})$$

and the morphisms

$$(A.2) \quad k_Q : K_Q \rightarrow h_{\mathbf{C}}(c)$$

for all squares $Q \in \mathbb{E}$ of the form (A.1). Here K_Q denotes the presheaf

$$K_Q = h_{\mathbf{C}}(c') \sqcup_{h_{\mathbf{C}}(d')} h_{\mathbf{C}}(d).$$

In particular, it follows that there exists an accessible localization functor $\mathcal{F} \rightarrow L_{\mathbb{E}}(\mathcal{F})$, left adjoint to the inclusion.

A.1.8. An \mathbb{E} -local equivalence is a morphism of presheaves which becomes invertible after applying the localization functor $L_{\mathbb{E}}$. The set of \mathbb{E} -local equivalences is equivalently the strongly saturated closure of the set of morphisms containing e and k_Q for each $Q \in \mathbb{E}$.

A.2. Topological excision structures.

A.2.1. Let \mathbb{E} be an excision structure on \mathbf{C} . We consider the following axioms on \mathbb{E} :

(EXC2) For every square of the form (A.1) in \mathbb{E} , the lower horizontal morphism f is a monomorphism.

(EXC3) For every square in \mathbb{E} of the form (A.1), the induced cartesian square

$$(A.3) \quad \begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow \\ d' \times_{c'} d' & \longrightarrow & d \times_c d, \end{array}$$

where the vertical morphisms are the diagonals, belongs to \mathbb{E} .

We define:

Definition A.2.2. An excision structure \mathbb{E} is topological if the axioms (EXC2) and (EXC3) are satisfied.

This terminology will be explained by Theorem A.2.9.

A.2.3. Let \mathbb{E} be an excision structure on \mathbf{C} .

Consider the Grothendieck pretopology on \mathbf{C} consisting of the following covering families:

- (1) The empty family covering $\emptyset_{\mathbf{C}}$.
- (2) For every square $Q \in \mathbb{E}$ of the form

$$\begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow g \\ c' & \xrightarrow{f} & c, \end{array}$$

the family $\{f, g\}$ covering c .

We let $\tau_{\mathbb{E}}$ denote the Grothendieck topology generated by this pretopology.

A.2.4. Note that the axioms (EXC0) and (EXC1) imply that families of the form (1) and (2) are stable under pullback.

It follows from [Hoy15a, Cor. C.2] that the condition of *descent* (Čech descent) with respect to the topology $\tau_{\mathbb{E}}$ can be described as follows.

Lemma A.2.5. A presheaf \mathcal{F} on \mathbf{C} is a $\tau_{\mathbb{E}}$ -sheaf if it satisfies the following conditions:

- (1) The space $\mathcal{F}(\emptyset_{\mathbf{C}})$ is contractible.
- (2) For every square $Q \in \mathbb{E}$ of the form (A.1), the canonical morphism of spaces

$$\mathcal{F}(c) \rightarrow \varinjlim_{[n] \in \Delta} \text{Maps}_{\text{PSh}(\mathbf{C})}(\check{C}(\tilde{c}/c)_n, \mathcal{F})$$

is invertible. Here the simplicial object $\check{C}(\tilde{c}/c)_\bullet$ is the Čech nerve of the morphism $\tilde{c} = h(c') \sqcup h(d) \rightarrow h(c)$.

A.2.6. Let $\mathbf{P}_{\tau_E}(\mathbf{C})$ denote the full sub- $(\infty, 1)$ -category of $\mathbf{PSh}(\mathbf{C})$ spanned by τ_E -sheaves, i.e. presheaves satisfying τ_E -descent.

Note that this is the left localization at the essentially small set containing the canonical morphism

$$e : \emptyset_{\mathbf{PSh}(\mathbf{C})} \rightarrow h_{\mathbf{C}}(\emptyset_{\mathbf{C}})$$

and the morphisms

$$(A.4) \quad c_Q : C_Q \rightarrow h_{\mathbf{C}}(c),$$

for all squares $Q \in \mathbb{E}$ of the form (A.1), where C_Q denotes the presheaf

$$C_Q = \varinjlim_{[n] \in \Delta^{op}} \check{C}(\tilde{c}/c)_n,$$

where $\tilde{c} = h(c') \sqcup h(d)$.

In particular, there is a localization functor $\mathcal{F} \mapsto L_{\tau}(\mathcal{F})$, left adjoint to the inclusion. By ∞ -topos theory, we have:

Proposition A.2.7. (i) *The localization functor L_{τ_E} is left exact, i.e. it commutes with finite limits.*

(ii) *The $(\infty, 1)$ -category $\mathbf{Sh}_{\tau_E}(\mathbf{C})$ has universality of colimits.*

A τ_E -local equivalence is a morphism of presheaves which becomes invertible after applying the localization functor L_{τ_E} . The set of τ_E -local equivalences is equivalently the strongly saturated closure of the set of morphisms containing e and c_Q for each $Q \in \mathbb{E}$.

A.2.8. The following theorem of Voevodsky says that the localization defined by any topological excision structure coincides with the localization defined by the associated Grothendieck topology.

Theorem A.2.9 (Voevodsky). *If \mathbb{E} is a topological excision structure, then for any presheaf \mathcal{F} on \mathbf{C} , the condition of \mathbb{E} -excision is equivalent to τ_E -descent.*

This was proved in [Voe10, Thm. 5.10] (cf. [AHW15, Thm. 3.2.5]) in the case where the $(\infty, 1)$ -category \mathbf{C} is an ordinary category. The proof generalizes *mutatis mutandis* to our setting.

In particular we obtain:

Corollary A.2.10. *If \mathbb{E} is a topological excision structure, then we have:*

(i) *The localization functor L_E is left-exact, i.e. commutes with finite limits.*

(ii) *The $(\infty, 1)$ -category $\mathbf{PSh}_E(\mathbf{C})$ has universality of colimits.*

A.3. Homotopy invariance.

A.3.1. Let \mathbf{C} be an essentially small $(\infty, 1)$ -category. Let \mathbb{A} be an (essentially small) set of morphisms in \mathbf{C} , which is stable under base change. That is, for each morphism $a : c' \rightarrow c$ in \mathbb{A} , and any morphism $f : d \rightarrow c$ in \mathbf{C} , the base change $c' \times_c d \rightarrow d$ exists and belongs to \mathbb{A} .

We define:

Definition A.3.2. A presheaf \mathcal{F} is \mathbb{A} -invariant if for every morphism $a : c' \rightarrow c$, the induced morphism of spaces

$$\mathcal{F}(a) : \mathcal{F}(c) \rightarrow \mathcal{F}(c')$$

is invertible.

Note that we have:

Lemma A.3.3. The condition of \mathbb{A} -invariance is stable by colimits.

Proof. Let \mathcal{F} be the colimit of a diagram $(\mathcal{F}_\alpha)_\alpha$, where each \mathcal{F}_α is \mathbb{A} -invariant. For any morphism $a \in \mathbb{A}$, the morphism $\mathcal{F}(a)$ is the colimit of the morphisms $\mathcal{F}_\alpha(a)$, since colimits of presheaves are computed object-wise. \square

A.3.4. Let $\mathrm{PSh}_{\mathbb{A}}(\mathbf{C})$ denote the full sub- $(\infty, 1)$ -category of $\mathrm{PSh}(\mathbf{C})$ spanned by presheaves that are \mathbb{A} -invariant. This is the left localization at the small set \mathbb{A} , and as such there exists a localization functor $\mathcal{F} \mapsto L_{\mathbb{A}}(\mathcal{F})$, left adjoint to the inclusion.

We say that a morphism of presheaves is an \mathbb{A} -local equivalence if it becomes invertible after applying the functor $L_{\mathbb{A}}$. The set of \mathbb{A} -local equivalences is equivalently the strongly saturated closure of the set \mathbb{A} .

A.3.5. According to [Hoy15b, Prop. 3.3], the basic properties of $\mathrm{PSh}_{\mathbb{A}}(\mathbf{C})$ can be summarized as follows:

Proposition A.3.6. (i) For each presheaf \mathcal{F} on \mathbf{C} , there is a canonical isomorphism

$$(A.5) \quad L_{\mathbb{A}}(\mathcal{F})(c) = \varinjlim_{(d \rightarrow c) \in (\mathbb{A}_c)^{\mathrm{op}}} \mathcal{F}(d)$$

for each object $c \in \mathbf{C}$, where \mathbb{A}_c denotes the full sub- $(\infty, 1)$ -category of $\mathbf{C}_{/c}$ spanned by compositions of morphisms in the set \mathbb{A} . Further, the category $(\mathbb{A}_c)^{\mathrm{op}}$ is sifted.

(ii) The functor $L_{\mathbb{A}}$ commutes with finite products.

(iii) The category $\mathrm{PSh}_{\mathbb{A}}(\mathbf{S})$ has universality of colimits.

A.4. Unstable homotopy theory.

A.4.1. Let \mathbf{C} be an essentially small $(\infty, 1)$ -category, admitting an initial object $\emptyset_{\mathbf{C}}$.

We fix an excision structure \mathbb{E} on \mathbf{C} , and an (essentially small) set \mathbb{A} of morphisms in \mathbf{C} , which is stable under base change.

The *unstable homotopy theory* associated to the pair (\mathbb{E}, \mathbb{A}) is the $(\infty, 1)$ -category $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$ defined as the full sub- $(\infty, 1)$ -category of $\mathrm{PSh}(\mathbf{C})$ spanned by presheaves satisfying \mathbb{E} -excision and \mathbb{A} -invariance.

A.4.2. This is a left localization, so $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$ is an ∞ -arena, and the inclusion admits an accessible left adjoint $\mathcal{F} \mapsto L_{\mathbb{E}, \mathbb{A}}(\mathcal{F})$.

A (\mathbb{E}, \mathbb{A}) -local equivalence is a morphism of presheaves that becomes invertible after applying $L_{\mathbb{E}, \mathbb{A}}$.

A.4.3. For an object $c \in \mathbf{C}$, we write

$$(A.6) \quad h_{\mathbf{C}}^{\mathbb{E}, \mathbb{A}}(c) = L_{\mathbb{E}, \mathbb{A}}(h_{\mathbf{C}}(c))$$

for the (\mathbb{E}, \mathbb{A}) -localization of the presheaf represented by c . Note that we have canonical bifunctorial isomorphisms of spaces

$$\mathrm{Maps}_{\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})}(h_{\mathbf{C}}^{\mathbb{E}, \mathbb{A}}(c), \mathcal{F}) = \mathcal{F}(c)$$

for any \mathbb{E} -excisive \mathbb{A} -invariant presheaf \mathcal{F} , by the Yoneda lemma and adjunction.

A.4.4. Recall that $\mathrm{PSh}(\mathbf{C})$ is the free ∞ -arena generated by the $(\infty, 1)$ -category \mathbf{C} .

More precisely, for any ∞ -arena \mathbf{S} , let $\mathrm{Funct}_!(\mathrm{PSh}(\mathbf{C}), \mathbf{S})$ denote the $(\infty, 1)$ -category of morphisms of ∞ -arenas $\mathrm{PSh}(\mathbf{C}) \rightarrow \mathbf{S}$. Then we have the following universal property:

Lemma A.4.5. *For any ∞ -arena \mathbf{S} , the canonical functor*

$$(A.7) \quad \mathrm{Funct}_!(\mathrm{PSh}(\mathbf{C}), \mathbf{S}) \rightarrow \mathrm{Funct}(\mathbf{C}, \mathbf{S}),$$

given by restriction along the Yoneda embedding $h_{\mathbf{C}} : \mathbf{C} \hookrightarrow \mathrm{PSh}(\mathbf{C})$, is an equivalence of $(\infty, 1)$ -categories.

In particular, any functor $u : \mathbf{C} \rightarrow \mathbf{S}$ admits a unique extension to a morphism of ∞ -arenas $u_! : \mathrm{PSh}(\mathbf{C}) \rightarrow \mathbf{S}$ (this is called the left Kan extension of u).

A.4.6. For an ∞ -arena \mathbf{S} , let $\mathrm{Funct}_{\mathbb{E}, \mathbb{A}}(\mathbf{C}, \mathbf{S})$ denote the full subcategory of $\mathrm{Funct}(\mathbf{C}, \mathbf{S})$ spanned by functors $u : \mathbf{C} \rightarrow \mathbf{S}$ that satisfy \mathbb{E} -excision and \mathbb{A} -invariance, i.e. which send

- (1) the initial object $\emptyset_{\mathbf{C}}$ to an initial object of \mathbf{S} ,
- (2) any square $Q \in \mathbb{E}$ to a cocartesian square in \mathbf{S} ,
- (3) and any morphism $a \in \mathbb{A}$ to an invertible morphism in \mathbf{S} .

We have the following universal property for the ∞ -arena $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$:

Proposition A.4.7. *For any ∞ -arena \mathbf{S} , the canonical functor*

$$(A.8) \quad \mathrm{Funct}_!(\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C}), \mathbf{S}) \rightarrow \mathrm{Funct}_{\mathbb{E}, \mathbb{A}}(\mathbf{C}, \mathbf{S}),$$

given by restriction along the functor $\mathbf{C} \rightarrow \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$, is an equivalence of $(\infty, 1)$ -categories.

Proof. This follows immediately from the universal properties of presheaf categories and of their left localizations. \square

A.4.8. Though the localization functors $L_{\mathbb{E}}$ and $L_{\mathbb{A}}$ do not commute, the functor $L_{\mathbb{E}, \mathbb{A}}$ can be described by the following transfinite composition:

Proposition A.4.9. *For every presheaf \mathcal{F} on \mathbf{C} , there is a canonical isomorphism*

$$(A.9) \quad L_{\mathbb{E}, \mathbb{A}}(\mathcal{F}) = \varinjlim_{n \geq 0} (L_{\mathbb{A}} \circ L_{\mathbb{E}})^{\circ n}(\mathcal{F}).$$

Proof. This follows from the fact that the properties of \mathbb{E} -excision and \mathbb{A} -invariance are stable by filtered colimits (Lemmas A.1.6 and A.3.3). \square

A.4.10. We have:

Corollary A.4.11. *If the excision structure \mathbb{E} is topological, then the following hold.*

- (i) *The localization functor $L_{\mathbb{E}, \mathbb{A}}$ commutes with finite products.*
- (ii) *The $(\infty, 1)$ -category $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$ has universality of colimits.*

Proof. The first claim follows from the formula given in Proposition A.4.9. Indeed, the functors $L_{\mathbb{E}}$ and $L_{\mathbb{A}}$ both commute with finite products (Lemmas A.2.10 and A.3.6), and filtered colimits commute with finite products of presheaves.

The second claim follows directly from Corollary A.2.10 and Proposition A.3.6. \square

As a consequence of (i), we have:

Corollary A.4.12. *If the excision structure \mathbb{E} is topological, then the cartesian monoidal structure on the ∞ -arena $\mathbf{PSh}(\mathbf{C})$ restricts to the ∞ -arena $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$.*

A.5. Pointed homotopy theory.

A.5.1. Let $\mathbf{PSh}(\mathbf{C})_{\bullet}$ denote the ∞ -arena of pointed presheaves, i.e. pointed objects in the ∞ -arena $\mathbf{PSh}(\mathbf{C})$. Hence its objects are pairs (\mathcal{F}, x) , where \mathcal{F} is a presheaf and $x : \mathrm{pt} \rightarrow \mathcal{F}$ is a morphism from the terminal presheaf.

By [Lur16a, Ex. 4.8.1.20, Prop. 4.8.2.11], $\mathbf{PSh}(\mathbf{C})_{\bullet}$ has a canonical structure of Spc_{\bullet} -module arena, and is canonically equivalent to the base change $\mathbf{PSh}(\mathbf{C}) \otimes_{\mathrm{Spc}} \mathrm{Spc}_{\bullet}$.

A.5.2. Consider the forgetful functor sending a pointed presheaf (\mathcal{F}, x) to its underlying presheaf \mathcal{F} . This admits a left adjoint, which freely adjoins a point to \mathcal{F} ; that is, it is given on objects by the assignment

$$\mathcal{F} \mapsto \mathcal{F}_+ := (\mathcal{F} \sqcup \mathrm{pt}, x)$$

where x is the canonical morphism $\mathrm{pt} \rightarrow \mathcal{F} \sqcup \mathrm{pt}$.

A.5.3. Note that $\mathbf{PSh}(\mathbf{C})_{\bullet}$ is equivalent to the category of modules over the monad with underlying endofunctor $\mathcal{F} \mapsto \mathcal{F} \sqcup \mathrm{pt}$. Since the latter commutes with contractible colimits, it follows that the forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$ is conservative, and preserves and reflects contractible colimits.

This monadic description also implies that every pointed presheaf can be written as the geometric realization of a simplicial diagram with each term in the essential image of $\mathcal{F} \mapsto \mathcal{F}_+$:

Lemma A.5.4. *The $(\infty, 1)$ -category $\mathbf{PSh}(\mathbf{C})_{\bullet}$ is generated under sifted colimits by objects of the form \mathcal{F}_+ , where \mathcal{F} is a presheaf on \mathbf{C} .*

A.5.5. The cartesian monoidal structure on the ∞ -arena $\mathbf{PSh}(\mathbf{S})$ induces a monoidal structure on $\mathbf{PSh}(\mathbf{C})_{\bullet}$ (see [Rob15, Cor. 2.32]):

Lemma A.5.6. *The ∞ -arena $\mathbf{PSh}(\mathbf{C})_{\bullet}$ admits a canonical symmetric monoidal structure, which is uniquely characterized by the fact that the functor $\mathcal{F} \mapsto \mathcal{F}_+$ is symmetric monoidal. Further, we have the following universal property:*

Given any symmetric monoidal morphism of ∞ -arenas $u : \mathbf{PSh}(\mathbf{C}) \rightarrow \mathbf{S}$, with \mathbf{S} a pointed ∞ -arena, there exists a unique symmetric monoidal morphism of ∞ -arenas $\tilde{u} : \mathbf{PSh}(\mathbf{C})_{\bullet} \rightarrow \mathbf{S}$, and an isomorphism $\tilde{u} \circ (-)_+ = u$.

We will write $\wedge_{\mathbf{PSh}(\mathbf{C})_{\bullet}}$ for the monoidal product, and $\underline{\mathrm{Hom}}_{\mathbf{PSh}(\mathbf{C})_{\bullet}}$ for the internal hom. The monoidal unit $\mathbf{1}_{\mathbf{PSh}(\mathbf{C})_{\bullet}}$ is given by pt_+ .

A.5.7. Let (\mathcal{F}, x) be a pointed presheaf, i.e. a presheaf \mathcal{F} with a morphism $x : \text{pt} \rightarrow \mathcal{F}$ from the terminal presheaf. We define:

Definition A.5.8. *The pointed presheaf (\mathcal{F}, x) is \mathbb{E} -excisive or \mathbb{A} -invariant if its underlying presheaf \mathcal{F} has the respective property.*

A.5.9. Let $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$ denote the full sub- $(\infty, 1)$ -category of $\mathbf{PSh}(\mathbf{C})_\bullet$ spanned by \mathbb{E} -excisive \mathbb{A} -invariant pointed presheaves. Note that this is equivalent to the ∞ -arena of pointed objects in the ∞ -arena $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$.

We call this the *pointed homotopy theory* associated to the pair (\mathbb{E}, \mathbb{A}) .

A.5.10. The monadic description we have given of $\mathbf{PSh}(\mathbf{C})_\bullet$ also applies to $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$, so that in particular we have:

Lemma A.5.11. *The forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$, on the $(\infty, 1)$ -category $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$, is conservative, and preserves and reflects contractible colimits.*

Similarly:

Lemma A.5.12. *The $(\infty, 1)$ -category $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$ is generated under sifted colimits by objects of the form \mathcal{F}_+ , where \mathcal{F} is an object of $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$.*

A.5.13. The symmetric monoidal structure on the ∞ -arena $\mathbf{PSh}(\mathbf{C})_\bullet$ restricts to one on $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$, uniquely characterized by the fact that the morphism $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C}) \rightarrow \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$ is symmetric monoidal.

As in Lemma A.5.6 we have the following universal property:

Lemma A.5.14. *Given any symmetric monoidal morphism of ∞ -arenas $u : \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C}) \rightarrow \mathbf{S}$, with \mathbf{S} a pointed ∞ -arena, there exists a unique symmetric monoidal morphism $\tilde{u} : \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet \rightarrow \mathbf{S}$, and an isomorphism $\tilde{u} \circ (-)_+ \approx u$.*

A.5.15. An immediate consequence of Proposition A.4.7 is the following universal property for the pointed ∞ -arena $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$:

Proposition A.5.16. *For any pointed ∞ -arena \mathbf{S} , the canonical functor*

$$(A.10) \quad \text{Funct}_!(\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet, \mathbf{S}) \rightarrow \text{Funct}_{\mathbb{E}, \mathbb{A}}(\mathbf{C}, \mathbf{S}),$$

given by restriction along the functor $\mathbf{C} \rightarrow \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$, is an equivalence of $(\infty, 1)$ -categories.

A.5.17. Note that $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_\bullet$ is the left localization at the essentially small set of morphisms of the form

$$(A.11) \quad h_{\mathbf{C}}(a)_+ : h_{\mathbf{C}}(c)_+ \rightarrow h_{\mathbf{C}}(c')_+,$$

for each morphism $a : c \rightarrow c'$ in \mathbb{A} ,

$$(A.12) \quad (k_{\mathbf{Q}})_+ : (K_{\mathbf{Q}})_+ \rightarrow h_{\mathbf{C}}(c)_+,$$

for each square $\mathbf{Q} \in \mathbb{E}$ of the form (A.1), and

$$e_+ : \text{pt} \rightarrow h_{\mathbf{C}}(\emptyset_{\mathbf{C}})_+.$$

A.5.18. In particular, we obtain localization functors $L_{\mathbb{E}}$, $L_{\mathbb{A}}$, and

$$(A.13) \quad L_{\mathbb{E}, \mathbb{A}} : \mathrm{PSh}(\mathbf{C})_{\bullet} \rightarrow \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_{\bullet}$$

at the level of pointed objects.

According to the universal property in Lemma A.5.6, these localization functors are symmetric monoidal morphisms of ∞ -arenas, characterized by commutativity with the functor $\mathcal{F} \mapsto \mathcal{F}_+$. This means we have canonical isomorphisms

$$L_{\mathbb{E}, \mathbb{A}}(h_{\mathbf{C}}(c)_+) = h_{\mathbf{C}}^{\mathbb{E}, \mathbb{A}}(c)_+$$

for each object $c \in \mathbf{C}$.

A.5.19. Let \mathcal{T} be a pointed presheaf belonging to the set \mathbb{T} .

The \mathcal{T} -suspension endofunctor $\Sigma_{\mathcal{T}}$ on $\mathrm{PSh}(\mathbf{C})_{\bullet}$ is defined by the assignment

$$(\mathcal{F}, x) \mapsto (\mathcal{F}, x) \wedge \mathcal{T}.$$

Dually, the \mathcal{T} -loop space endofunctor $\Omega_{\mathcal{T}}$ is given by

$$(\mathcal{F}, x) \mapsto \underline{\mathrm{Hom}}(\mathcal{T}, (\mathcal{F}, x)).$$

These endofunctors form an adjunction $(\Sigma_{\mathcal{T}}, \Omega_{\mathcal{T}})$.

A.6. Stable homotopy theory.

A.6.1. We fix an (essentially small) set $\mathbb{T}_{\mathbf{C}}$ of pointed presheaves on \mathbf{C} .

We will always assume that each $\mathcal{T} \in \mathbb{T}_{\mathbf{C}}$ is \mathbb{E} -excisive and \mathbb{A} -invariant, by replacing it with its (\mathbb{E}, \mathbb{A}) -localization if necessary.

Consider the following axioms:

(STAB1) Each pointed presheaf $\mathcal{T} \in \mathbb{T}$ is k -symmetric, i.e. the cyclic permutation of $\mathcal{T}^{\otimes k}$ is homotopic to the identity morphism, for some $k \geq 2$.

(STAB2) At least one of the pointed presheaves $\mathcal{T} \in \mathbb{T}$ can be written as an \mathbf{S}^1 -suspension $\Sigma_{\mathbf{S}^1}(\mathcal{T}')$ of some pointed presheaf \mathcal{T}' .

A.6.2. Let $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ denote the ∞ -arena of \mathbb{T} -spectrum objects in the ∞ -arena $\mathrm{PSh}(\mathbf{C})_{\bullet}$. We briefly recall its construction from [Hoy15b, §6.1].

Suppose that the set \mathbb{T} contains a finite number of elements $\mathcal{T}_1, \dots, \mathcal{T}_n$. Then let $\mathcal{T} = \mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_n$, and define $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ as the colimit of the filtered diagram

$$(A.14) \quad \mathrm{PSh}(\mathbf{C})_{\bullet} \xrightarrow{\Sigma_{\mathcal{T}}} \mathrm{PSh}(\mathbf{C})_{\bullet} \xrightarrow{\Sigma_{\mathcal{T}}} \dots$$

in the $(\infty, 1)$ -category of ∞ -arenas. Equivalently, this is the limit of the cofiltered diagram

$$(A.15) \quad \dots \xrightarrow{\Omega_{\mathcal{T}}} \mathrm{PSh}(\mathbf{C})_{\bullet} \xrightarrow{\Omega_{\mathcal{T}}} \mathrm{PSh}(\mathbf{C})_{\bullet}$$

in the $(\infty, 1)$ -category of $(\infty, 1)$ -categories.

In general, we define $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ as the filtered colimit

$$\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) = \varinjlim_{\mathbb{T}_0 \subset \mathbb{T}} \mathrm{Spt}_{\mathbb{T}_0}(\mathrm{PSh}(\mathbf{C})_{\bullet}),$$

where the colimit is indexed over finite subsets $\mathbb{T}_0 \subset \mathbb{T}$.

Remark A.6.3. Note that, in the case where \mathbb{T} is finite, a \mathbb{T} -spectrum is the data of a sequence $(\mathcal{F}_n)_{n \geq 0}$ of pointed presheaves and structural isomorphisms

$$\alpha_n : \mathcal{F}_n \xrightarrow{\sim} \Omega_{\mathbb{T}}(\mathcal{F}_{n+1})$$

for each integer $n \geq 0$.

In general, we have, roughly speaking, deloopings with respect to any finite tensor product of elements of \mathbb{T} , and a homotopy coherent system of compatibilities.

Example A.6.4. Take \mathbb{T} to be the set containing only the constant pointed presheaf \mathbf{S}^1 . Then the ∞ -arena $\mathrm{Spt}_{\mathbf{S}^1}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ is canonically equivalent to the ∞ -arena of presheaves of spectra on \mathbf{C} .

A.6.5. By construction, the adjunction $(\Sigma_{\mathbb{T}}, \Omega_{\mathbb{T}})$ at the level of pointed spaces gives rise to an equivalence

$$\Sigma_{\mathbb{T}} : \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) \rightleftarrows \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) : \Omega_{\mathbb{T}}.$$

A.6.6. Also by construction, we have for each $n \geq 0$ a canonical functor

$$\Omega_{\mathbb{T}}^{\infty-n} : \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) \rightarrow \mathrm{PSh}(\mathbf{C})_{\bullet},$$

with canonical isomorphisms $\Omega_{\mathbb{T}} \Omega_{\mathbb{T}}^{\infty-n-1} = \Omega_{\mathbb{T}}^{\infty-n}$.

Dually we have a canonical functor

$$\Sigma_{\mathbb{T}}^{\infty-n} : \mathrm{PSh}(\mathbf{C})_{\bullet} \rightarrow \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}),$$

left adjoint to $\Omega_{\mathbb{T}}^{\infty-n}$, with canonical isomorphisms $\Sigma_{\mathbb{T}}^{\infty-n-1} \Sigma_{\mathbb{T}}^{\infty-n} = \Sigma_{\mathbb{T}}^{\infty-n}$.

We will write $\Omega_{\mathbb{T}}^{\infty} := \Omega_{\mathbb{T}}^{\infty-0}$ and $\Sigma_{\mathbb{T}}^{\infty} := \Sigma_{\mathbb{T}}^{\infty-0}$. Note that there are canonical isomorphisms $\Sigma_{\mathbb{T}}^{\infty-n} = \Omega_{\mathbb{T}}^n \Sigma_{\mathbb{T}}^{\infty}$ and $\Omega_{\mathbb{T}}^{\infty-n} = \Omega_{\mathbb{T}}^n \Sigma_{\mathbb{T}}^{\infty}$ for each n .

A.6.7. We have canonical bifunctorial isomorphisms of spaces

$$(A.16) \quad \mathrm{Maps}_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}(\Sigma_{\mathbb{T}}^{\infty}(\mathrm{h}_{\mathbf{C}}(c)_+), \mathcal{F}) = \Omega_{\mathbb{T}}^{\infty}(\mathcal{F})(c).$$

for every object $c \in \mathbf{C}$ and every \mathbb{T} -spectrum \mathcal{F} .

A.6.8. Since a pointed $(\infty, 1)$ -category is stable if and only if the \mathbf{S}^1 -suspension functor $\Sigma_{\mathbf{S}^1}$ is an equivalence, we have:

Lemma A.6.9. *If the axiom (STAB2) holds, then the ∞ -arenas $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ is stable.*

A.6.10. By [Lur09a, Lem. 6.3.3.6], and the construction of $\mathrm{Spt}_{\mathbb{T}}(\mathbf{S})$, we have:

Lemma A.6.11. *The category $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ is generated under filtered colimits by objects of the form $\Sigma_{\mathbb{T}}^{\infty-n}(\mathcal{F})$, for \mathcal{F} a pointed presheaf on \mathbf{C} and $n \geq 0$.*

A.6.12. Suppose that the axiom (STAB1) holds, i.e. each pointed presheaf $\mathcal{F} \in \mathbb{T}$ is k -symmetric for some $k \geq 2$. Then the main result of [Rob15] endows the ∞ -arena $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ with a canonical symmetric monoidal structure:

Lemma A.6.13. *If the axiom (STAB1) holds, then the ∞ -arena $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ admits a canonical symmetric monoidal structure, and the functor $\Sigma_{\mathbb{T}}^{\infty}$ lifts to a symmetric monoidal morphism of ∞ -arenas, which sends each $\mathcal{F} \in \mathbb{T}$ to a monoidally invertible object of $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$.*

Further, it satisfies the following universal property: for any symmetric monoidal ∞ -arena \mathbf{S} , and any symmetric monoidal morphism $u : \mathrm{PSh}(\mathbf{C})_{\bullet} \rightarrow \mathbf{S}$ sending each $\mathcal{F} \in \mathbb{T}$ to a monoidally invertible object in \mathbf{S} , there exists a unique symmetric monoidal morphism $\tilde{u} : \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) \rightarrow \mathbf{S}$ and an isomorphism $\tilde{u} \circ \Sigma_{\mathbb{T}}^{\infty} = u$.

We will write $\otimes_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}$ for the monoidal product in $\mathrm{Spt}_{\mathbb{T}}(\mathbf{S})$, and $\underline{\mathrm{Hom}}_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}$ for the internal hom. The monoidal unit $\mathbf{1}_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}$ is the \mathbb{T} -spectrum $\Sigma_{\mathbb{T}}^{\infty}(\mathrm{pt}_{+})$.

A.6.14. We will abuse notation and write \mathcal{T} also for the monoidally invertible object $\Sigma_{\mathbb{T}}^{\infty}(\mathcal{T})$, for each $\mathcal{T} \in \mathbb{T}$. Let $\mathcal{T}^{\otimes 0} = \mathbf{1}$ and let $\mathcal{T}^{\otimes(-1)}$ be a monoidal inverse to \mathcal{T} ; for $n > 1$, write $\mathcal{T}^{\otimes(-n)} = (\mathcal{T}^{\otimes(-1)})^{\otimes n}$.

The universal property in Lemma A.6.13 shows that there are canonical isomorphisms of functors $(\Sigma_{\mathcal{T}})^{\circ n} = \mathcal{T}^{\otimes n} \otimes (-)$ and $(\Omega_{\mathcal{T}})^{\circ n} = \mathcal{T}^{\otimes -n} \otimes (-)$ for each $n \geq 0$.

A.6.15. We define:

Definition A.6.16. *We say that a \mathbb{T} -spectrum \mathcal{F} satisfies \mathbb{E} -excision or \mathbb{A} -invariance if for each $n \geq 0$, its n th component $\Omega_{\mathbb{T}}^{\infty - n}(\mathcal{F})$ satisfies the respective property (as a pointed presheaf).*

Let $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$ denote the full sub- $(\infty, 1)$ -category of $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ spanned by \mathbb{E} -excisive \mathbb{A} -invariant \mathbb{T} -spectra. This is equivalent to the ∞ -arena of $\mathrm{L}_{\mathbb{E}, \mathbb{A}}(\mathbb{T})$ -spectrum objects in the ∞ -arena $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_{\bullet}$, where $\mathrm{L}_{\mathbb{E}, \mathbb{A}}(\mathbb{T})$ denotes the set $\{\mathrm{L}_{\mathbb{E}, \mathbb{A}}(\mathcal{T})\}_{\mathcal{T} \in \mathbb{T}}$.

We call the ∞ -arena $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$ the *stable homotopy theory* associated to $(\mathbb{E}, \mathbb{A}, \mathbb{T})$.

A.6.17. As in Lemma A.6.9, we have:

Lemma A.6.18. *If the axiom (STAB2) holds, then the ∞ -arena $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}$ is stable.*

A.6.19. We have canonical bifunctorial isomorphisms of spaces

$$(A.17) \quad \mathrm{Maps}_{\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})}(\Sigma_{\mathbb{T}}^{\infty}(\mathrm{h}_{\mathbf{C}}^{\mathbb{E}, \mathbb{A}}(c)_{+}), \mathcal{F}) = \Omega_{\mathbb{T}}^{\infty}(\mathcal{F})(c).$$

for every object $c \in \mathbf{C}$ and every \mathbb{E} -excisive \mathbb{A} -invariant \mathbb{T} -spectrum \mathcal{F} .

A.6.20. As in Lemma A.6.11, we have:

Lemma A.6.21. *The category $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$ is generated under filtered colimits by objects of the form $\Sigma_{\mathbb{T}}^{\infty - n}(\mathcal{F})$, where \mathcal{F} is a pointed presheaf belonging to $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_{\bullet}$ and $n \geq 0$.*

A.6.22. It is possible to extend the universal property of $\mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_{\bullet}$ (Proposition A.5.16) to formulate a universal property describing functors out of $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$.

Lemma A.6.23. *Let $u : \mathbf{C} \rightarrow \mathbf{S}$ be a functor to a pointed ∞ -arena \mathbf{S} which satisfies \mathbb{E} -excision and \mathbb{A} -invariance. Then u can be lifted to a functor $u^{\infty} : \mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C}) \rightarrow \mathbf{S}$ if and only if there exists a sequence of functors $(u_n)_{n \geq 0}$, with $u = u_0$, together with isomorphisms $u_{n+1} \circ \Sigma_{\mathbb{T}} \rightarrow u_n$.*

This is just the universal property of the colimit defining $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$.

We will provide another universal property below, in terms of symmetric monoidal structures.

A.6.24. Under the axiom (STAB1), the symmetric monoidal structure on the ∞ -arena $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$ (Lemma A.6.13) restricts to one on $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$. Further, we have the following universal property, just as in Lemma A.6.13:

Lemma A.6.25. *If the axiom (STAB1) holds, then the ∞ -arena $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$ admits a canonical symmetric monoidal structure, and the functor*

$$\Sigma_{\mathbb{T}}^{\infty} : \mathbf{H}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})_{\bullet} \rightarrow \mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$$

lifts to a symmetric monoidal morphism of ∞ -arenas, which sends each $\mathcal{T} \in \mathbb{T}$ to a monoidally invertible object of $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$.

Further, it satisfies the following universal property: for any symmetric monoidal ∞ -arena \mathbf{S} , and any symmetric monoidal morphism $u : \mathrm{PSh}(\mathbf{C})_{\bullet} \rightarrow \mathbf{S}$ sending each $\mathcal{T} \in \mathbb{T}$ to a monoidally invertible object in \mathbf{S} , there exists a unique symmetric monoidal morphism $\tilde{u} : \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) \rightarrow \mathbf{S}$ and an isomorphism $\tilde{u} \circ \Sigma_{\mathbb{T}}^{\infty} = u$.

We will write $\otimes_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}$ for the monoidal product in $\mathrm{Spt}_{\mathbb{T}}(\mathbf{S})$, and $\underline{\mathrm{Hom}}_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}$ for the internal hom. The monoidal unit $\mathbf{1}_{\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})}$ is the \mathbb{T} -spectrum $\Sigma_{\mathbb{T}}^{\infty}(\mathrm{pt}_{+})$.

A.6.26. For a pointed symmetric monoidal ∞ -arena \mathbf{S} , let $\mathrm{Funct}_{!}^{\otimes}(\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C}), \mathbf{S})$ denote the $(\infty, 1)$ -category of morphisms of symmetric monoidal ∞ -arenas $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C}) \rightarrow \mathbf{S}$.

Let $\mathrm{Funct}^{\otimes}(\mathbf{C}, \mathbf{S})$ denote the $(\infty, 1)$ -category of symmetric monoidal functors $\mathbf{C} \rightarrow \mathbf{S}$, and $\mathrm{Funct}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}^{\otimes}(\mathbf{C}, \mathbf{S})$ the full subcategory spanned by symmetric monoidal morphisms $u : \mathbf{C} \rightarrow \mathbf{S}$ that satisfy \mathbb{E} -excision, \mathbb{A} -invariance (see (A.4.6)), and the condition that, for each pointed presheaf $\mathcal{T} \in \mathbb{T}$, the cofibre $\mathrm{Cofib}(\mathrm{pt} \rightarrow \mathcal{T})$ is sent to a monoidally invertible object of \mathbf{S} .

We obtain the following universal property for the ∞ -arena $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$, as a corollary of the various universal properties that we have already obtained:

Corollary A.6.27. *For any pointed symmetric monoidal ∞ -arena \mathbf{S} , the canonical functor*

$$(A.18) \quad \mathrm{Funct}_{!}^{\otimes}(\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C}), \mathbf{S}) \rightarrow \mathrm{Funct}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}^{\otimes}(\mathbf{C}, \mathbf{S}),$$

given by restriction along the canonical functor $\mathbf{C} \rightarrow \mathbf{SH}_{\mathbb{E}, \mathbb{A}}(\mathbf{C})$, is an equivalence of $(\infty, 1)$ -categories.

A.6.28. Note that $\mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C})$ is the left localization at the essentially small set of morphisms of the form

$$(A.19) \quad \Sigma_{\mathbb{T}}^{\infty} h_{\mathbf{C}}(a)_{+} : \Sigma_{\mathbb{T}}^{\infty} h_{\mathbf{C}}(c)_{+} \rightarrow \Sigma_{\mathbb{T}}^{\infty} h_{\mathbf{C}}(c')_{+},$$

for each morphism $a : c \rightarrow c'$ in \mathbb{A} ,

$$(A.20) \quad \Sigma_{\mathbb{T}}^{\infty} (k_Q)_{+} : \Sigma_{\mathbb{T}}^{\infty} (K_Q)_{+} \rightarrow \Sigma_{\mathbb{T}}^{\infty} h_{\mathbf{C}}(c)_{+},$$

for each square $Q \in \mathbb{E}$ of the form (A.1), and

$$\Sigma_{\mathbb{T}}^{\infty} (e_{+}) : 0 \rightarrow \Sigma_{\mathbb{T}}^{\infty} h_{\mathbf{C}}(\emptyset_{\mathbf{C}})_{+},$$

where 0 denotes the zero object of $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet})$.

In particular, we have localization functors $L_{\mathbb{E}}$, $L_{\mathbb{A}}$, and

$$(A.21) \quad L_{\mathbb{E}, \mathbb{A}} : \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_{\bullet}) \rightarrow \mathbf{SH}_{\mathbb{E}, \mathbb{A}, \mathbb{T}}(\mathbf{C}),$$

at the level of \mathbb{T} -spectra.

Using the universal property (Lemma A.6.13), we see that they define morphisms of symmetric monoidal ∞ -arenas which can be characterized uniquely by commutativity with the functor $\Sigma_{\mathbb{T}}^{\infty}$. In particular, we have

$$L_{\mathbb{E}, \mathbb{A}}(\Sigma_{\mathbb{T}}^{\infty - k}(h_{\mathbf{C}}(c)_{+})) = \Sigma_{\mathbb{T}}^{\infty - k}(h_{\mathbf{C}}^{\mathbb{E}, \mathbb{A}}(c)_{+})$$

for each $k \geq 0$.

APPENDIX B. COEFFICIENT SYSTEMS

In this section we will introduce a formalism for working with categories of coefficients. These are systems of categories $\mathbf{D}(\mathbf{S})$, indexed by objects in some category \mathbf{C} (e.g. schemes), equipped with some basic functorialities f^* and f_* associated to any morphism f in \mathbf{C} . This is a natural framework for stating base change and projection formulas.

We will use the language of $(\infty, 2)$ -categories as a convenient way to formulate some of the results in this section, even though we only apply them in the $(\infty, 2)$ -category of ∞ -arenas. Following [GR16], our definition of $(\infty, 2)$ -category will be a complete Segal space in the $(\infty, 1)$ -category of $(\infty, 1)$ -categories.

B.1. Passage to right/left adjoints.

B.1.1. Recall that in an $(\infty, 2)$ -category \mathbb{C} , there is a notion of adjunction between two objects x and y .

A pair $(f : x \rightarrow y, g : y \rightarrow x)$ forms an adjunction if and only if it defines an adjunction in the underlying ordinary 2-category $(\mathbb{C})^{2\text{-ordn}}$.

B.1.2. Let \mathbb{C} and \mathbb{D} be $(\infty, 2)$ -categories. We say that an $(\infty, 2)$ -functor $u : \mathbb{C} \rightarrow \mathbb{D}$ is *right-adjointable* (resp. *left-adjointable*) if, for each 1-morphism $f : x \rightarrow y$ in \mathbb{C} , its image $u(f)$ admits a right adjoint (resp. a left adjoint) in the 2-category \mathbb{D} .

B.1.3. Let $\text{Maps}_!(\mathbb{S}, \mathbb{T})$ denote the space of right-adjointable $(\infty, 2)$ -functors. Let $\text{Maps}_*(\mathbb{S}, \mathbb{T})$ denote the space of left-adjointable $(\infty, 2)$ -functors.

Lemma B.1.4. *There is a canonical isomorphism of spaces*

$$\text{Maps}_!(\mathbb{S}, \mathbb{T}) = \text{Maps}_*((\mathbb{S})^{1\&2\text{-op}}, \mathbb{T}),$$

where $(\mathbb{C})^{1\&2\text{-op}}$ denotes the $(\infty, 2)$ -category obtained by flipping the directions of 1- and 2-morphisms in \mathbb{S} .

See [GR16, Cor. 1.3.4].

Given a right-adjointable $(\infty, 2)$ -functor $u : \mathbb{S} \rightarrow \mathbb{T}$, we will call the corresponding functor $u_* : (\mathbb{S})^{1\&2\text{-op}} \rightarrow \mathbb{T}$ the functor obtained from u by *passage to right adjoints*.

Dually, given a left-adjointable $(\infty, 2)$ -functor $u : \mathbb{S} \rightarrow \mathbb{T}$, we will call the corresponding functor $u_! : (\mathbb{S})^{1\&2\text{-op}} \rightarrow \mathbb{T}$ the functor obtained from u by *passage to left adjoints*.

B.2. Adjointable squares. In this section we will formulate the notion of *horizontally/vertically left/right-adjointable square* in a 2-category, which we will be used in the text to express base change formulas.

B.2.1. We fix an $(\infty, 2)$ -category \mathbb{C} .

Let Θ be a square in \mathbb{C}

$$(B.1) \quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{u} & \mathbf{C}' \\ \downarrow v & & \downarrow v' \\ \mathbf{D} & \xrightarrow{u'} & \mathbf{D}' \end{array}$$

which commutes up to an invertible 2-morphism

$$v'u \xrightarrow{\sim} u'v.$$

Suppose that v (resp. v') admits a right adjoint v^R (resp. $(v')^R$) in \mathbf{C} . Then the square $\Theta^{vert:R}$

$$(B.2) \quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{u} & \mathbf{C}' \\ v^R \uparrow & & \uparrow (v')^R \\ \mathbf{D} & \xrightarrow{u'} & \mathbf{D}' \end{array}$$

commutes up to the 2-morphism

$$(B.3) \quad uv^R \rightarrow (v')^R v' u v^R \xrightarrow{\sim} (v')^R u' v v^R \rightarrow (v')^R u',$$

where the first morphism is obtained by precomposition with the counit of the adjunction, the isomorphism in the middle is given by the commutativity of the square Θ , and the final morphism is given by the unit of the adjunction.

If this 2-morphism is invertible, then we say that the square Θ is *vertically right-adjointable*.

B.2.2. Similarly if v (resp. v') admits a *left* adjoint v^L (resp. $(v')^L$), then the square $\Theta^{vert:L}$

$$(B.4) \quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{u} & \mathbf{C}' \\ v^L \uparrow & & \uparrow (v')^L \\ \mathbf{D} & \xrightarrow{u'} & \mathbf{D}' \end{array}$$

commutes up to the 2-morphism

$$(B.5) \quad (v')^L u' \rightarrow (v')^L u' v v^L \xleftarrow{\sim} (v')^L v' u v^L \rightarrow u v^L.$$

If this is invertible, we say that the square Θ is *vertically left-adjointable*.

B.2.3. If u (resp. u') admits a right adjoint u^R (resp. $(u')^R$), then the square $\Theta^{horiz:R}$

$$(B.6) \quad \begin{array}{ccc} \mathbf{C} & \xleftarrow{u^R} & \mathbf{C}' \\ \downarrow v & & \downarrow v' \\ \mathbf{D} & \xleftarrow{(u')^R} & \mathbf{D}' \end{array}$$

commutes up to a 2-morphism

$$(B.7) \quad v u^R \rightarrow (u')^R v'.$$

If it is invertible, we say that Θ is *horizontally right-adjointable*.

Similarly, if u (resp. u') admits a left adjoint u^L (resp. $(u')^L$), then the square $\Theta^{horiz:L}$

$$(B.8) \quad \begin{array}{ccc} \mathbf{C} & \xleftarrow{u^L} & \mathbf{C}' \\ \downarrow v & & \downarrow v' \\ \mathbf{D} & \xleftarrow{(u')^L} & \mathbf{D}' \end{array}$$

commutes up to a 2-morphism

$$(B.9) \quad (u')^L v' \rightarrow v u^L.$$

If it is invertible, we say that Θ is *horizontally left-adjointable*.

B.2.4. We have:

Lemma B.2.5. *Suppose that in the square Θ (B.1), u (resp. u') admits a right adjoint u^R (resp. $(u')^R$), and v (resp. v') admits a left adjoint v^L (resp. $(v')^L$). Then Θ is vertically left-adjointable if and only if it is horizontally right-adjointable.*

Proof. The square $\Theta^{vert:L}$ (resp. $\Theta^{horiz:R}$) commutes up to a 2-morphism $\alpha : (v')^L u' \rightarrow uv^L$ (resp. $\beta : vu^R \rightarrow (u')^R v'$). The category of left adjoint functors $\mathbf{C} \rightarrow \mathbf{D}$ is equivalent to the category of right adjoint functors $\mathbf{D} \rightarrow \mathbf{C}$ (see [GR16, Chap. A.3, Cor. 3.1.9]), and under this equivalence the morphism α corresponds to the morphism β . \square

B.3. Coefficient systems. Let *Arena* denote the symmetric monoidal $(\infty, 1)$ -category of arenas. Recall that an arena is an accessible left localization of a presheaf category $\mathbf{PSh}(\mathbf{C})$ with \mathbf{C} small, and a morphism of arenas is a colimit-preserving functor.

Let *Arenamon* denote the $(\infty, 1)$ -category of symmetric monoidal arenas, which are by definition commutative monoids in *Arena*. Note that these are nothing else than symmetric monoidal categories whose underlying category is an arena, such that the monoidal product commutes with colimits in each argument.

For the rest of this section, we fix an $(\infty, 1)$ -category \mathbf{C} which admits fibred products. Note that the cartesian monoidal structure on \mathbf{C} induces a canonical symmetric monoidal structure on $(\mathbf{C})^{\text{op}}$.

B.3.1.

Definition B.3.2. *A coefficient system (defined on \mathbf{C}) is a symmetric monoidal functor*

$$\mathbf{D}^* : (\mathbf{C})^{\text{op}} \rightarrow \text{Arena}.$$

Given a coefficient system \mathbf{D}^* , we will write

$$\mathbf{D}(S) := \mathbf{D}^*(S)$$

for the arena associated to an object $S \in \mathbf{C}$. We will write \emptyset_S (resp. e_S) for the initial (resp. terminal) object of $\mathbf{D}(S)$. We will often refer to the objects of $\mathbf{D}(S)$ as *sheaves* on S .

For a morphism $f : T \rightarrow S$, we will write

$$f^* := \mathbf{D}^*(f) : \mathbf{D}(S) \rightarrow \mathbf{D}(T)$$

for the induced functor, which we call the functor of *inverse image* along f . It is cocontinuous, and admits (by the adjoint functor theorem) a right adjoint

$$f_* : \mathbf{D}(T) \rightarrow \mathbf{D}(S)$$

which we call the functor of *direct image* along f .

B.3.3. By passing to right adjoints (see Paragraph B.1), \mathbf{D}^* gives rise to a unique functor

$$\mathbf{D}_* : \mathbf{C} \rightarrow (\infty, 1)\text{-Cat}$$

such that each functor

$$\mathbf{D}_*(f) : \mathbf{D}(T) \rightarrow \mathbf{D}(S)$$

is the right adjoint f_* .

B.3.4. Since the functor \mathbf{D}^* underlying a coefficient system is symmetric monoidal, it sends cocommutative comonoids in \mathbf{C} to commutative monoids in Arena. Note that every object $S \in \mathbf{C}$ has a canonical structure of cocommutative comonoid (with respect to the cartesian monoidal structure).

Hence for each object $S \in \mathbf{C}$, the arena $\mathbf{D}(S)$ has a canonical symmetric monoidal structure, and for each morphism f in \mathbf{C} , the inverse image functor f^* has a canonical symmetric monoidal structure (giving by adjunction a lax monoidal structure on its right adjoint f_*).

We will write \otimes_S for the monoidal product of $\mathbf{D}(S)$, 1_S for the monoidal unit, and $\underline{\text{Hom}}_S$ for the internal hom.

B.3.5. Dually, suppose we are given a functor (not necessarily symmetric monoidal)

$$\mathbf{D}_! : \mathbf{C} \rightarrow \text{Arena}.$$

We will write $\mathbf{D}(S) := \mathbf{D}_!(S)$ for the arena associated to an object $S \in \mathbf{C}$. For each morphism $f : T \rightarrow S$ in \mathbf{C} , we write

$$f_! := \mathbf{D}_!(f) : \mathbf{D}(T) \rightarrow \mathbf{D}(S)$$

for the induced functor, which commutes with colimits and admits a right adjoint $f^!$.

As above, we can pass to right adjoints to obtain a functor $\mathbf{D}^! : (\mathbf{C})^{\text{op}} \rightarrow \text{Arena}$.

B.3.6. For future use, we make the following definition:

Definition B.3.7. *A coefficient system \mathbf{D}^* is pointed (resp. \mathbf{S}^1 -stable, compactly generated) if the functor $\mathbf{D}^* : (\mathbf{C})^{\text{op}} \rightarrow \text{Arena}$ factors through the full subcategory spanned by pointed (resp. stable, compactly generated) arenas.*

B.4. Left-adjointability.

B.4.1. Let us fix a class *left* of *left-admissible* morphisms in \mathbf{C} , containing all isomorphisms, closed under composition and base change, and satisfying the 2-out-of-3 property. Let \mathbf{C}^{left} denote the (non-full) subcategory of \mathbf{C} spanned by left-admissible morphisms.

Definition B.4.2. *We say that the coefficient system \mathbf{D}^* is weakly left-adjointable along a morphism $p : T \rightarrow S$ if it satisfies the following property:*

(Adj^p) *The functor p^* admits a left adjoint p_\sharp .*

We say that \mathbf{D}^ is weakly left-adjointable along the class *left* if it satisfies the following property:*

(Adj^{left}) *For every left-admissible morphism p , the property (Adj^p) holds.*

Note that if \mathbf{D}^* is weakly left-adjointable along *left*, then one obtains a canonical functor

$$\mathbf{D}_\sharp : \mathbf{C}^{\text{left}} \rightarrow \text{Arena}$$

by passage to left adjoints (see Paragraph B.1).

B.4.3. Recall the notion of adjointability of squares from Paragraph B.2.

Definition B.4.4. *The coefficient system \mathbf{D}^* satisfies left base change along a morphism $p : S' \rightarrow S$ if it is weakly left-adjointable along *left*, and the following property holds:*

(BC^p) For all cartesian squares Θ

$$(B.10) \quad \begin{array}{ccc} T' & \xrightarrow{f'} & S' \\ \downarrow p' & & \downarrow p \\ T & \xrightarrow{f} & S, \end{array}$$

the induced commutative square Θ^*

$$(B.11) \quad \begin{array}{ccc} \mathbf{D}(S) & \xrightarrow{f^*} & \mathbf{D}(T) \\ \downarrow p^* & & \downarrow (p')^* \\ \mathbf{D}(S') & \xrightarrow{(f')^*} & \mathbf{D}(T'). \end{array}$$

is vertically left-adjointable.

We say that \mathbf{D}^* satisfies left base change along the class left if it is weakly left-adjointable along left, and the following property holds:

(BC^{left}) For every left-admissible morphism p , the property (BC^p) holds.

In other words, \mathbf{D}^* satisfies left base change along a morphism p if for every such cartesian square Θ , the exchange 2-morphism

$$(B.12) \quad (p')_{\sharp}(f')^* \rightarrow f^* p_{\sharp}$$

is invertible.

According to Lemma B.2.5, this is equivalent to the condition that its right transpose

$$(B.13) \quad p^* f_* \rightarrow (f')_*(p')^*$$

is invertible.

B.4.5. For any morphism $f : T \rightarrow S$, the symmetric monoidal functor $f^* : \mathbf{D}(S) \rightarrow \mathbf{D}(T)$ gives $\mathbf{D}(T)$ a structure of $\mathbf{D}(S)$ -module category. If f_{\sharp} is left adjoint to f^* , then it admits a canonical structure of colax morphism of $\mathbf{D}(S)$ -modules. In particular there are canonical morphisms

$$f_{\sharp}(\mathcal{F} \otimes_T f^*(\mathcal{G})) \rightarrow f_{\sharp}(\mathcal{F}) \otimes_S \mathcal{G} \quad (\mathcal{F} \in \mathbf{D}(T), \mathcal{G} \in \mathbf{D}(S)).$$

Definition B.4.6. The coefficient system \mathbf{D}^* satisfies the left projection formula along a morphism $p : T \rightarrow S$ if it is weakly left-adjointable along p , and the following property holds:

(Proj^p) The colax morphism of $\mathbf{D}(S)$ -modules p_{\sharp} is strict.

We say that \mathbf{D}^* satisfies the left projection formula along the class left if it is weakly left-adjointable along left, and the following property holds:

(Proj^{left}) For every left-admissible morphism p , the property (Proj^{left}) holds.

In other words, \mathbf{D}^* satisfies the left projection formula along $p : T \rightarrow S$ if the canonical morphisms

$$(B.14) \quad p_{\sharp}(\mathcal{F} \otimes_T p^*(\mathcal{G})) \rightarrow p_{\sharp}(\mathcal{F}) \otimes_S \mathcal{G} \quad (\mathcal{F} \in \mathbf{D}(T), \mathcal{G} \in \mathbf{D}(S))$$

are invertible.

B.4.7.

Definition B.4.8. A coefficient system \mathbf{D}^* is left-adjointable along the class left if it is weakly left-adjointable along left (Adj^{left}), satisfies left base change along left (BC^{left}), and satisfies the left projection formula along left (Proj^{left}).

B.5. Right-adjointability.

B.5.1. Let us fix a class *right* of *right-admissible* morphisms in \mathbf{C} , containing all isomorphisms, and closed under composition and base change. Let \mathbf{C}^{right} denote the (non-full) subcategory of \mathbf{C} spanned by right-admissible morphisms.

B.5.2. The following definitions are dual to the definitions in Paragraph B.4.

Definition B.5.3. A coefficient system \mathbf{D}^* is weakly right-adjointable along a morphism q if it satisfies the following property:

(Adj_{*q*}) The direct image functor q_* admits a right adjoint.

We say that \mathbf{D}^* is weakly right-adjointable along a class *right* if it satisfies the following property:

(Adj_{*right*}) For each right-admissible morphism q , the property (Adj_{*q*}) holds.

Definition B.5.4. A coefficient system \mathbf{D}^* satisfies right base change along a morphism $q : S' \rightarrow S$ if the following property holds:

(BC_{*right*}) For all cartesian squares Θ in \mathbf{C}

$$\begin{array}{ccc} T' & \xrightarrow{f'} & S' \\ \downarrow q' & & \downarrow q \\ T & \xrightarrow{f} & S \end{array}$$

with q and q' right-admissible, the induced commutative square Θ^*

$$\begin{array}{ccc} \mathbf{D}(S) & \xrightarrow{f^*} & \mathbf{D}(T) \\ \downarrow q^* & & \downarrow (q')^* \\ \mathbf{D}(S') & \xrightarrow{(f')^*} & \mathbf{D}(T'). \end{array}$$

is vertically right-adjointable.

We say that \mathbf{D}^* satisfies right base change along the class *right* if the following property holds:

(BC_{*right*}) For every right-admissible morphism q , the property (BC_{*q*}) holds.

In other words, \mathbf{D}^* satisfies right base change along a morphism q if for every such cartesian square Θ , the exchange 2-morphism

$$f^* p_* \rightarrow (p')_*(f')^*$$

is invertible.

Definition B.5.5. A coefficient system \mathbf{D}^* satisfies the right projection formula along a morphism $q : T \rightarrow S$ if the following property holds:

(Proj_{*q*}) The canonical structure of lax morphism of $\mathbf{D}(S)$ -module arenas on q_* is strict.

We say that \mathbf{D}^* satisfies the right projection formula along the class *right* if the following property holds:

(Proj_{*right*}) For every right-admissible morphism q , the property (Proj_{*q*}) holds.

In other words, \mathbf{D}^* satisfies the right projection formula along $q : T \rightarrow S$ if the canonical morphisms

$$(B.15) \quad q_*(\mathcal{F} \otimes_T q^*(\mathcal{G})) \rightarrow q_*(\mathcal{F}) \otimes_S \mathcal{G} \quad (\mathcal{F} \in \mathbf{D}(T), \mathcal{G} \in \mathbf{D}(S)).$$

are invertible.

Definition B.5.6. A coefficient system \mathbf{D}^* is right-adjointable along the class *right* if it is weakly right-adjointable along *right* ($\text{Adj}_{\text{right}}$), satisfies right base change along *right* (BC_{right}), and satisfies the right projection formula along *right* ($\text{Proj}_{\text{right}}$).

B.6. Biadjointability.

B.6.1. Let \mathbf{D}^* be a coefficient system which is left-adjointable along *left*.

For every cartesian square Θ in \mathbf{C}

$$(B.16) \quad \begin{array}{ccc} T' & \xrightarrow{q'} & S' \\ \downarrow p' & & \downarrow p \\ T & \xrightarrow{q} & S \end{array}$$

with p and p' left-admissible, we have by left base change along *left* (BC^{left}) a commutative square

$$(B.17) \quad \begin{array}{ccc} \mathbf{D}(S) & \xrightarrow{q^*} & \mathbf{D}(T) \\ p_{\#} \uparrow & & (p')_{\#} \uparrow \\ \mathbf{D}(S') & \xrightarrow{(q')^*} & \mathbf{D}(T'). \end{array}$$

Suppose that \mathbf{D}^* is also right-adjointable along *right* and that q and q' are right-admissible. Then one can ask whether the above square is horizontally right-adjointable, i.e. whether the square

$$(B.18) \quad \begin{array}{ccc} \mathbf{D}(S) & \xleftarrow{q_*} & \mathbf{D}(T) \\ p_{\#} \uparrow & & (p')_{\#} \uparrow \\ \mathbf{D}(S') & \xleftarrow{(q')_*} & \mathbf{D}(T'). \end{array}$$

commutes via the 2-morphism

$$(B.19) \quad p_{\#}(q')_* \rightarrow q_* q^* p_{\#}(q')_* \simeq q_*(p')_{\#}(q')^*(q')_* \rightarrow q_*(p')_{\#}.$$

Definition B.6.2. The coefficient system \mathbf{D}^* satisfies bidirectional base change along the pair (*left*, *right*) if it satisfies left base change along *left*, right base change along *right*, and the following property holds:

($\text{BC}_{\text{right}}^{\text{left}}$) For all cartesian squares Θ in \mathbf{C}

$$\begin{array}{ccc} T' & \xrightarrow{q'} & S' \\ \downarrow p' & & \downarrow p \\ T & \xrightarrow{q} & S \end{array}$$

with p and p' left-admissible (resp. q and q' right-admissible), the square (B.18) commutes.

In other words, we require that for all cartesian squares Θ as above, the 2-morphism

$$(B.20) \quad p_{\#}(q')^* \rightarrow q_*(p')_{\#}.$$

is invertible.

According to Lemma B.2.5, this is equivalent to the condition that the right transpose

$$(B.21) \quad (p')^* q^! \rightarrow (q')^! p^*$$

is invertible.

B.6.3. Finally, we define:

Definition B.6.4. *A coefficient system \mathbf{D}^* is (left, right)-biadjointable if it left-adjointable along left, right-adjointable along right, and satisfies bidirectional base change along (left, right) $(\mathbf{BC}_{right}^{left})$.*

APPENDIX C. LOCALLY COCONTINUOUS FUNCTORS

In this section we introduce some technical topos-theoretic tools that are used in Sect. 6 to proof that the functor i_* commutes with contractible colimits.

C.1. Local cocontinuity. Recall that a functor u between sites is *topologically cocontinuous*⁶ if the restriction of presheaves functor u^* preserves local equivalences. In this paragraph we introduce a slightly weaker version of this condition, where the functor u is only “locally cocontinuous” with respect to a weaker topology. We will show that for such functors, the restriction functor u^* preserves local equivalences between sheaves for the weaker topology.

C.1.1. Let \mathbf{C} be an essentially small $(\infty, 1)$ -category. We will write $\mathbf{PSh}(\mathbf{C})$ for the $(\infty, 1)$ -category of presheaves on \mathbf{C} , and $h : \mathbf{C} \hookrightarrow \mathbf{PSh}(\mathbf{C})$ for the Yoneda embedding. Given a topology τ on \mathbf{C} , we will write $\mathbf{Sh}_{\tau}(\mathbf{C})$ for the subcategory of τ -sheaves, i.e. the presheaves \mathcal{F} for which the canonical morphism

$$\mathcal{F}(c) \rightarrow \text{Maps}(\mathbf{R}, \mathcal{F})$$

is invertible for all τ -covering sieves $\mathbf{R} \hookrightarrow h(c)$ of all objects $c \in \mathbf{C}$. We will write $\text{inc}_{\tau} : \mathbf{Sh}_{\tau}(\mathbf{C}) \hookrightarrow \mathbf{PSh}(\mathbf{C})$ for the inclusion, and $L_{\tau} : \mathbf{PSh}(\mathbf{C}) \rightarrow \mathbf{Sh}_{\tau}(\mathbf{C})$ for the left-exact left adjoint (the τ -localization functor).

Given a functor $u : \mathbf{C} \rightarrow \mathbf{D}$, we will write $u^* : \mathbf{PSh}(\mathbf{D}) \rightarrow \mathbf{PSh}(\mathbf{C})$ for the restriction of presheaves functor, and $u_!$ (resp. u_*) for the left adjoint (resp. right adjoint).

C.1.2. Let (\mathbf{C}, τ) and (\mathbf{D}, τ') be $(\infty, 1)$ -sites. Recall that a functor $u : \mathbf{C} \rightarrow \mathbf{D}$ is *topologically cocontinuous* if the following condition is satisfied:

(COC) For every τ' -covering sieve $\mathbf{R}' \hookrightarrow h(u(c))$, the sieve $\mathbf{R} \hookrightarrow h(c)$, generated by morphisms $c' \rightarrow c$ such that $h(u(c')) \rightarrow h(u(c))$ factors through \mathbf{R}' , is τ -covering.

Note that \mathbf{R} can be described as the sieve

$$(C.1) \quad \mathbf{R} = u^*(\mathbf{R}') \times_{u^*(h(u(c)))} h(c) \hookrightarrow h(c)$$

obtained from $\mathbf{R}' \hookrightarrow h(u(c))$ by applying u^* and taking the base change along the unit morphism $h(c) \rightarrow u^*u_!(h(c)) = u^*h(u(c))$.

⁶The term *cocontinuous* is used in [AGV73]. Nowadays this term is often used for colimit-preserving functors, so we have slightly modified the classical terminology in order to avoid confusion.

C.1.3. Let τ'_0 be a topology on \mathbf{D} which is weaker than τ' . For simplicity we will assume that τ'_0 is subcanonical, so that representable presheaves are τ'_0 -sheaves. Let $L_{\tau'_0}$ denote the associated τ'_0 -sheaf functor, left adjoint to the inclusion.

We will say that u is τ'_0 -*locally topologically cocontinuous* if it satisfies the following weaker version of the condition (COC):

(COC') For every τ' -covering sieve $R' \hookrightarrow h(u(c))$, the sieve $R \hookrightarrow h(c)$, generated by morphisms $c' \rightarrow c$ such that $h(u(c')) \rightarrow h(u(c))$ factors through the τ'_0 -sheaf associated to R' , is τ -covering.

Note that R can be described as the sieve

$$(C.2) \quad R := u^*(L_{\tau'_0}(R')) \times_{u^*(h(u(c)))} h(c) \hookrightarrow h(c)$$

obtained in the same way as (C.1) starting from $L_{\tau'_0}(R') \hookrightarrow h(u(c))$.

C.1.4. The following lemma is proved in exactly the same way as the analogous result for topologically cocontinuous functors [AGV73, Exp. III, Prop. 2.2]:

Lemma C.1.5. *Let $u : (\mathbf{C}, \tau) \rightarrow (\mathbf{D}, \tau')$ be a functor. Let τ'_0 be a topology on \mathbf{D} which is weaker than τ' . If u is τ'_0 -locally cocontinuous, then the functor*

$$u_0^* : \mathrm{Sh}_{\tau'_0}(\mathbf{D}) \hookrightarrow \mathrm{PSh}(\mathbf{D}) \xrightarrow{u^*} \mathrm{PSh}(\mathbf{C})$$

sends τ' -local equivalences to τ -local equivalences.

Proof. First of all, note that u_0^* admits a left adjoint

$$u_!^0 : \mathrm{PSh}(\mathbf{C}) \xrightarrow{u_!} \mathrm{PSh}(\mathbf{D}) \xrightarrow{L_{\tau'_0}} \mathrm{Sh}_{\tau'_0}(\mathbf{D})$$

by construction. Let $R' \hookrightarrow h(d)$ be a τ' -covering sieve of an object d of \mathbf{D} . To show that $\vartheta : u_0^*(R') \hookrightarrow u_0^*(h(d))$ is a τ -local equivalence, it suffices by universality of colimits to show that, for every object c of \mathbf{C} and every morphism $\varphi : h(c) \rightarrow u_0^*h(d)$, the base change

$$u_0^*R' \times_{u_0^*h(d)} h(c) \hookrightarrow h(c)$$

is a τ -covering sieve. Note that φ factors canonically through the unit morphism $h(c) \rightarrow u_0^*u_!^0h(c) = u_0^*h(u(c))$ and the canonical morphism $u_0^*u_!^0h(c) = u_0^*h(u(c)) \rightarrow u_0^*h(d)$ (obtained by adjunction from φ). The base change of ϑ by $u_0^*h(u(c)) \rightarrow u_0^*h(d)$ is identified, since u_0^* commutes with limits, with the canonical morphism

$$u^*(R' \times_{h(d)} h(u(c))) \rightarrow u_0^*h(u(c)).$$

Since $R' \times_{h(d)} h(u(c)) \hookrightarrow h(u(c))$ is τ' -covering, as the base change of a τ' -covering sieve, the conclusion follows by applying the condition (COC'). \square

C.2. The reduced topology. In this paragraph we consider a topology whose associated category of sheaves coincides with the free completion by contractible colimits. This topology, which we denote τ_{red} , is the one associated to the minimal excision structure, with no commutative squares (see Paragraph A.1 for the notion of excision structure). We show that for any τ_{red} -locally topologically cocontinuous functor u , the restriction functor u^* on sheaves commutes with contractible colimits.

C.2.1. Let \mathbf{C} be an $(\infty, 1)$ -category admitting an initial object $\emptyset_{\mathbf{C}}$. We let τ_{red} , the *reduced topology*, be the Grothendieck topology on \mathbf{C} defined by the single sieve $\emptyset \hookrightarrow h(\emptyset_{\mathbf{C}})$, where \emptyset is the initial presheaf.

Note that a presheaf $\mathcal{F} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Spc}$ is a τ_{red} -sheaf if and only if the space $\mathcal{F}(\emptyset_{\mathbf{C}})$ is contractible. We will refer to τ_{red} -sheaves as *reduced presheaves*.

C.2.2. Let $\text{Sh}_{\text{red}}(\mathbf{C})$ denote the $(\infty, 1)$ -category of reduced presheaves. It is not difficult to show that $\text{Sh}_{\text{red}}(\mathbf{C})$ is the $(\infty, 1)$ -category freely generated by \mathbf{C} under contractible colimits.

Let L_{red} denote the τ_{red} -localization functor, left adjoint to the inclusion $\text{inc}_{\text{red}} : \text{Sh}_{\text{red}}(\mathbf{C}) \hookrightarrow \text{PSh}(\mathbf{C})$. For a presheaf \mathcal{F} on \mathbf{C} , $L_{\text{red}}(\mathcal{F})$ can be described as the unique reduced presheaf for which the space $L_{\text{red}}(\mathcal{F})(c)$ is identified with $\mathcal{F}(c)$ whenever c is not initial.

C.2.3. For convenience we state an easy-to-use sufficient condition for τ_{red} -local cocontinuity.

Lemma C.2.4. *Let (\mathbf{C}, τ) and (\mathbf{D}, τ') be $(\infty, 1)$ -sites and $u : \mathbf{C} \rightarrow \mathbf{D}$ a functor. Assume that \mathbf{D} admits an initial object $\emptyset_{\mathbf{D}}$, and that the topology τ' is stronger than τ_{red} . Then for the functor u to be τ_{red} -locally cocontinuous, the following condition is sufficient:*

(COC'_{\tau_{\text{red}}}) For every τ' -covering sieve $R' \hookrightarrow h(u(c))$, the sieve $R \hookrightarrow h(c)$, generated by morphisms $c' \rightarrow c$ such that either $h(u(c')) \rightarrow h(u(c))$ factors through $R' \hookrightarrow h(u(c))$ or $u(c')$ is initial, is τ -covering.

Indeed let $c' \rightarrow c$ be a morphism such that $u(c')$ is initial. Then the (unique) morphism $h(u(c')) = L_{\text{red}}(\emptyset) \rightarrow h(u(c))$ factors as the composite of the (unique) morphism $L_{\text{red}}(\emptyset) \rightarrow L_{\text{red}}(R')$ and $L_{\text{red}}(R') \rightarrow h(u(c))$.

C.2.5. The following lemma is a formal consequence of Lemma C.1.5, and the fact that reduced presheaves are stable by contractible colimits:

Lemma C.2.6. *Suppose that u is τ_{red} -locally cocontinuous. Then the composite functor*

$$\text{Sh}_{\tau'}(\mathbf{D}) \xrightarrow{\text{inc}_{\tau'}} \text{PSh}(\mathbf{D}) \xrightarrow{u^*} \text{PSh}(\mathbf{C}) \xrightarrow{L_{\tau}} \text{Sh}_{\tau}(\mathbf{C})$$

commutes with contractible colimits.

Proof. Since the topology τ' is a refinement of τ_{red} , the inclusion $\text{inc}_{\tau'}$ factors as

$$\text{inc}_{\tau'} : \text{Sh}_{\tau'}(\mathbf{D}) \xrightarrow{\text{inc}_{\tau'}^{\text{red}}} \text{Sh}_{\text{red}}(\mathbf{D}) \xrightarrow{\text{inc}^{\text{red}}} \text{PSh}(\mathbf{D}),$$

and its left adjoint $L_{\tau'}$ factors as

$$L_{\tau'} : \text{PSh}(\mathbf{D}) \xrightarrow{L^{\text{red}}} \text{Sh}_{\tau'_{\text{red}}}(\mathbf{D}) \xrightarrow{L_{\tau'}^{\text{red}}} \text{Sh}_{\tau'}(\mathbf{D}),$$

and $L_{\tau'}^{\text{red}}$ is left adjoint to $\text{inc}_{\tau'}^{\text{red}}$.

Given a diagram $(\mathcal{F}_i)_{i \in \mathbf{I}}$ of τ' -sheaves indexed by a contractible $(\infty, 1)$ -category \mathbf{I} , consider the counit morphism

$$\varinjlim_i \text{inc}_{\tau'}^{\text{red}}(\mathcal{F}_i) \longrightarrow \text{inc}_{\tau'}^{\text{red}} L_{\tau'}^{\text{red}} \varinjlim_i \text{inc}_{\tau'}^{\text{red}}(\mathcal{F}_i),$$

which is clearly a τ' -local equivalence. By applying $u_{\text{red}}^* = u^* \text{inc}_{\text{red}}$ this induces a morphism

$$u_{\text{red}}^* \varinjlim_i \text{inc}_{\tau'}^{\text{red}}(\mathcal{F}_i) \longrightarrow u_{\text{red}}^* \text{inc}_{\tau'}^{\text{red}} L_{\tau'}^{\text{red}} \varinjlim_i \text{inc}_{\tau'}^{\text{red}}(\mathcal{F}_i),$$

which is identified with a canonical morphism

$$\varinjlim_i u^* \text{inc}_{\tau'}(\mathcal{F}_i) \longrightarrow u^* \text{inc}_{\tau} \varinjlim_i \mathcal{F}_i$$

since the inclusion inc_{red} commutes with contractible colimits. By Lemma C.1.5, this is a τ -local equivalence, so the claim follows. \square

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